JOSEPH S. ALPER and MARK BRIDGER

MATHEMATICS, MODELS AND ZENO'S PARADOXES*

ABSTRACT. A version of nonstandard analysis, Internal Set Theory, has been used to provide a resolution of Zeno's paradoxes of motion. This resolution is inadequate because the application of Internal Set Theory to the paradoxes requires a model of the world that is not in accordance with either experience or intuition. A model of standard mathematics in which the ordinary real numbers are defined in terms of rational intervals does provide a formalism for understanding the paradoxes. This model suggests that in discussing motion, only intervals, rather than instants, of time are meaningful. The approach presented here reconciles resolutions of the paradoxes based on considering a finite number of acts with those based on analysis of the full infinite set Zeno seems to require. The paper concludes with a brief discussion of the classical and quantum mechanics of performing an infinite number of acts in a finite time.

1. INTRODUCTION

For almost 2500 years, Zeno's paradoxes of motion have attracted the interest of philosophers, mathematicians, and scientists (see, Salmon 1970 and Vlastos 1967 for textual and historical background). Recently McLaughlin and Miller (hereafter, MM) published a new resolution of the paradoxes (McLaughlin and Miller 1992; see also, McLaughlin 1994). They argue that the mathematical formalism used in previous resolutions of the paradoxes is inadequate and should be replaced by a recent version of nonstandard analysis, Internal Set Theory (IST). In this paper; we critically analyze MM's solution and offer our suggestions for supplementing the classic work of Vlastos and Grünbaum (Vlastos 1967; and Grünbaum 1968).

In Section 2, we review Zeno's paradoxes and the Simple Mathematical Theory (SMT), to borrow MM's label, used to resolve them. We then outline, in Section 3, the criticisms of the naive use of this mathematical formalism. Section 4 summarizes the philosophical discussions of Vlastos and Grünbaum, which we believe provide the most satisfactory attempts to overcome these criticisms within the context of SMT. The following section provides an introduction to Internal Set Theory. In Section 6, we first present MM's application of this theory in their resolution of the paradoxes and then argue that their resolution is not helpful. In essence we suggest that a resolution based on IST is even more paradoxical than Zeno's

Synthese **110:** 143–166, 1997.

paradoxes themselves. In Sections 7 and 8 we present our suggestions for resolving the paradoxes. We argue that concepts of time and motion are best understood by regarding motion as a more fundamental concept than time. Consequently, time is defined in terms of motion. Using this definition, it is shown that time should be analyzed in terms of intervals rather than instants. The paradoxes are then discussed using a model of the ordinary real number system based on intervals. This approach unifies the positions taken by Vlastos and Grünbaum and provides additional justification for some of their assertions. In Section 9, we re-examine the question of whether an infinite number of tasks (to be defined below) can be performed in a finite time. If it is assumed that physical measurements can be made with arbitrary accuracy, then such a performance is not impossible according to the laws of classical mechanics, but is prohibited by the Heisenberg uncertainty principle of quantum mechanics. Section 10 summarizes our results.

2. ZENO'S PARADOXES

(Z1) This paradox, often referred to as the "Race Course", or the "Dichotomy", claims to show that a runner can never complete a race. In order to do so, the runner must first traverse 1/2 the distance, then the next 1/4 of the distance, etc. In addition, even if all these decreasing intervals are traversed, the runner can not reach the end point of the course.

(Z2) The second paradox, also referred to as the Dichotomy, appears to show that not only can the runner not complete the race, but cannot even begin it. To reach the end of the race course, the runner must first reach the half-way point. However, this entails first reaching the quarter-way point, etc.

The most famous paradox, known as the "Achilles", shows that if Achilles gives a tortoise a head start in a race, Achilles can never catch up to the tortoise. By the time Achilles reaches the point where the tortoise was when the race started, the tortoise has already moved some distance. Achilles now runs this shorter distance, but the tortoise has continued to progress during this shorter time interval, etc. By looking at the race from the viewpoint of the tortoise, this paradox can be essentially reduced to the Race Course (Z1), so we need not discuss it explicitly.

THE ARROW. This paradox appears to demonstrate that motion is impossible. At any given instant of time, an arrow in flight occupies a definite position in space, an interval precisely equal to its length. If at this and at every other instant of time it occupies a single region of space, it is not moving at these instants. Then, when does it move?

In order to speak precisely about paradoxes (Z1) and (Z2), it is customary to take the race course to be the unit interval [0, 1] in R, the real numbers. The runner is regarded as a point, moving at constant velocity equal to 1, which traverses the interval in one unit of time. If $\phi(t)$ denotes the position of the point at time t, then $\phi(t) = t$.

From the perspective of SMT, which is based on calculus and infinite series, Zeno's paradoxes seem unproblematic. The distance traveled at any given time is given by the (differentiable) function $\phi(t)$. The Arrow is resolved by noting that at any instant of time, τ, the object is moving with a velocity equal to the derivative $\phi'(\tau)$. To resolve (Z1), note that the times needed to traverse the various subintervals of [0, 1] form the convergent infinite series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots$ whose limit is 1. Thus the total distance can be completed in a time interval of length 1, which seems to eliminate the objection that the infinitely many subintervals would take infinitely long to traverse. (Z2) is resolved by an analogous argument.

The discussion given in the preceding paragraph is based on several assumptions concerning the relevance and applicability of the mathematical model to actual physical motion. Contemporary interest in the paradoxes lies in the analysis of the use of R to model space and time and the use of infinite series in describing a physical process.

3. OBJECTIONS TO THE USE OF THE SIMPLE MATHEMATICAL THEORY

There are at least four objections to the purely mathematical resolution of the paradoxes. These objections are central to philosophical as well as scientific analyses of Zeno's work.

First, the application of SMT to (Z1) and (Z2) is inadequate because it entails the performance of an infinite number of acts, represented mathematically by the summing of an infinite series or physically by the traversal of all the segments of the path. The possibility of performing an infinite number of acts has been rejected by many philosophers since classical times. The oldest reason for this rejection is that an infinite number of acts can not be performed in a finite amount of time. SMT avoids this problem by assuming a physical model in which time elapsed is proportional to distance covered and time, like distance, is itself infinitely divisible. This model, and hence SMT's validity, was rejected by various philosophers who denied that time could be modeled by the real number continuum

(James 1948; and Whitehead 1929). These philosophers argued that “the temporal order of occurrence of physical events is isomorphic with the discrete order of the nows of our awareness” (Grünbaum 1968, 48). Further, each of these “nows of our awareness” requires a minimal length of time. Thus, performing an infinitude of acts would take infinitely long. We believe that Grünbaum (1968) showed successfully that even though there may be a minimum length of time required for the perception of an event, within the context of a scientific description, time is infinitely divisible. (See also, Smart 1967 for a mathematical treatment of becoming and other perceptual aspects of time.)

Other philosophers, while not subscribing to the view that the perceptual nature of time cannot be captured mathematically, still regard the notion of performing infinitely many acts as inherently meaningless or self-contradictory (see, for example, Thomson 1967a). Although Thomson admitted that his argument was refuted by Benacerraf (1967), he maintained that Benacerraf could not show that the performance of an infinite number of events in a finite time is not inherently meaningless or self-contradictory (Thomson 1967b). We will discuss this issue in more detail below.

Second, in the case of (Z1), even though the sum of the infinite series is equal to 1, the union of the infinite set of intervals $[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}]$, n = 1, 2 ... does not include the endpoint 1 of [0, 1].

Third, in the case of (Z2), the motion would have to be initiated during the time corresponding to the “last term” in the sequence of times $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ .... Since there is no last term, SMT fails to account for the initiation of the motion.

One can argue that SMT does not actually prescribe performing an infinite number of acts, but rather replaces this completed infinity with that masterpiece of 19th century mathematics, the notion of *limit.* This raises the question: In what sense does a formal mathematical construct such as limit explain an observation about the physical world? As we shall see, this question applies no less to MM’s resolution of Zeno’s paradoxes using Internal Set Theory than it does to SMT.

Fourth, with regard to the Arrow, although the derivative can be used to represent speed at an instant, its relevance in explaining motion at that instant is problematic. In order to compute the derivative at a particular time τ, we must be able to evaluate ϕ at all t in some (open) neighborhood of τ. Thus, defining the motion of the arrow using the derivative at a particular time τ requires knowledge about the arrow’s position at additional instants of time; i.e. we must know its motion throughout some interval containing τ.

4. PHILOSOPHICAL RESOLUTIONS OF ZENO'S PARADOXES

In their fundamental inquiry, Vlastos and Grünbaum offered resolutions of Zeno's paradoxes within the framework of SMT (Vlastos 1967; and Grünbaum 1968). Both deal with the Arrow by observing that velocity and hence motion can only be defined in an interval of time and that it is meaningless to talk about the motion at a single instant of time. Vlastos resolves (Z1) and (Z2) by arguing that Zeno's partition of the interval [0, 1] into infinitely many pieces is not necessary in order to account for its traversal. Grünbaum, on the other hand, does believe that it is necessary to consider this partition, but argues that completing an infinite number of tasks (traversing all the subintervals) in a finite time is both possible and intelligible.

Vlastos defines a Z*-interval* to be one of the intervals $[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]$, and a Z*-run* to be the traversal of one or more contiguous Z-intervals. He then makes an important distinction. If a Z-run is to be considered as a single "physically individuated" event, he refers to it as a Z_a-run. On the other hand, a Z_b-run is one for which "*the traversal of any subinterval we please* by a runner would also count as a 'run' " (Vlastos 1967, 373). Thus, a Z_b-run might be considered as partitioned into 2 sub-runs (e.g., by the midpoint), 10 sub-runs, or even an infinite number of sub-runs by considering, e.g., all the Z-intervals. (We should note that Vlastos actually uses the terms *run*$_a$ and *run*$_b$. We find the "Z_a- and Z_b-run" terminology a natural extension of the idea of a "Z-run" and, in addition, more euphonious.)

Vlastos points out that Z_a-runs are the ones we normally think of as separate events or actions, while Zeno has cleverly used the word "run" in both senses. On one hand, he plays on our willingness to conceive of infinite subdivisions (Z_b-runs) and, on the other hand, on our unwillingness to admit the possibility of infinitely many discrete acts (Z_a-runs). Zeno's argument can now be rephrased.

- [0, 1] can be partitioned into the collection of all Z-runs (Z_b sense).
- Performing each such Z-run (Z_a sense) is necessary in order to traverse [0, 1].
- Since each Z_a-run is an act, this violates the proscription against a completed infinity of actions.

If we are willing to consider the run [0, 1] in the Z_b sense, then there is no problem with *describing* it in terms of any number of (Z_b) runs. In fact, there are infinitely many different ways of so describing it. However, as Vlastos points out, no particular description or decomposition is necessary in order to describe the run [0, 1]; in fact, any particular description is sufficient, including the description consisting of [0, 1] itself: "all that is needed to consume $\aleph_0$ parts (parts$_b$) of an egg is simply to eat an egg" (Vlastos 1967, 373).

Thus, Zeno's second assertion is suspect, since breaking down the uniform motion of the runner into an infinite number of Z_a-runs is neither semantically honest nor, in fact, what we physically do when we observe the runner. Here is where Zeno traps us by shifting from a Z_b to a Z_a interpretation of the runs. We essentially agree with Vlastos' analysis and will supply below some further details concerning the justification for the distinction between Z_a-runs and Z_b-runs.

Vlastos' analysis shows that we are not *forced* to consider an infinite number of actions. However, the question concerning the *possibility* of performing an infinite number of them remains. For a discussion of this issue, we turn to Grünbaum's analysis of "infinity machines" (Grünbaum 1968). Grünbaum argues that it is "kinematically" possible to conceive of machines that perform an infinite number of delineated acts in a finite time. By kinematically possible, Grünbaum means that the motion involves no discontinuities or infinities in position or velocity. He gives the example of the "staccato runner" who runs twice as fast as the usual runner for the first half of each Z-interval, then rests for the remainder. The rests serve to delineate the motions. In this first version of the example, such a runner expends an infinite amount of energy. In a later paper, using a construction of Friedberg, Grünbaum shows how this motion can be smoothed (Grünbaum 1970). The smoothed motion involves an infinite sequence of sub-motions, each terminating in a finite rest period. The interval is traversed in one unit of time. This motion violates no law of classical (as opposed to quantum) physics (see Section 9). If one grants that such motions are of type Z_a, then Grünbaum has resolved Zeno's problem by showing that the performance of infinitely many acts does not seem to violate any logical or physical laws.

Grünbaum also proposes a solution to the problem of how the runner can reach the point 1 even though no Z-interval contains 1 (Grünbaum 1968). He notes that the completion of the infinite number of subintervals is "equivalent" to reaching 1 since the length of the interval [0, 1) is equal to the length of the interval [0, 1]. Vlastos had previously given a similar argument, also based on distances (Vlastos 1967, 374). We believe that neither answer is convincing. An alternative explanation based on the nature of the real number system is given below in Section 8.

In their discussion of Zeno's paradoxes, MM do not directly refer to the arguments of Vlastos and Grünbaum. Nevertheless, because these earlier arguments rely on the framework of SMT, MM presumably find them inadequate. MM assert that a resolution of Zeno's paradoxes can be accomplished by using mathematics that explicitly introduces infinitesimals in a rigorous fashion. In order to appreciate and criticize their analysis, we need to examine some results of nonstandard analysis on which their work is based.

5. INTERNAL SET THEORY

In the early 1960's, the logician Abraham Robinson showed that "first order" statements about the reals have a nonstandard model in which "infinitesimals" and "infinitely large" numbers exist (Robinson 1966). Such models have been constructed (see Steen 1971 for a survey). In these models, an infinitesimal is a number which is bigger than 0 and smaller than any "standard" positive real. An infinite real can be defined as the reciprocal of an infinitesimal. It should be emphasized that this construction is a nonstandard model for the *usual* axiomatization of the reals. Nonstandard models for the reals are possible because only first-order statements about real numbers are required to be true in a model. The completeness of the reals, as expressed for example by the Least Upper Bound property, is not a first order statement and, indeed, does not hold in any model of the reals that contains infinitesimals (Steen 1971).

In 1977, Nelson showed how infinitesimals could be obtained by the addition of certain axioms to the usual Zermelo-Fraenkel set theory (Nelson 1977). Because of the addition of these axioms, his approach, Internal Set Theory (IST), can be considered to be a change in the fundamental structure of mathematics. In IST the existence of infinitesimals is guaranteed by a theorem, hence they must appear in every model of the theory. By contrast, Robinson's infinitesimals exist only in certain (nonstandard) models of Zermelo-Fraenkel set theory.

First, Nelson introduces an undefined property (predicate) called *standard*. An object for which this predicate is true is called a standard object. Intuitively, the standard objects include all the constructed objects of classical set theory. Formulas which do not involve the predicate "standard" are said to be *internal*; the others are called *external*. Nelson now adds three axioms governing the behavior of this predicate. We list these axioms below, with a brief explanation of each, and encourage the reader to consult Nelson's elegant paper for the complete theory (Nelson, 1977).

We follow Nelson's notation by using $\forall^{\mathrm{st}}x$ and $\exists^{\mathrm{st}}x$ to denote "For all x, x standard implies ..." and "There is an x such that x is standard and ...", respectively. $\forall^{\mathrm{fin}}x$ and $\exists^{\mathrm{fin}}x$ are defined similarly for the predicate *finite*. In mathematics, a set is finite if and only if it can not be placed in

1–1 correspondence with a proper subset of itself. However, in the context of Internal Set Theory, this definition does not always coincide with the usual intuitive meaning of the word "finite". This point is crucial in our critique of MM's discussion of Zeno.

(I) *Idealization.* Let $B(x, y)$ be an *internal* formula with free variables x, y and possibly other free variables. Then

$$\forall^{\text{st fin}} z \exists x \forall y \in z \quad B(x,y) \Leftrightarrow \exists x \forall^{\text{st}} y \quad B(x,y).$$

This axiom is extremely powerful. It implies that a relation has a simultaneous solution over *all* standard sets if and only if it has a solution for every finite standard set. Suppose B is the statement "x is a non-zero positive real smaller than all positive reals y". Clearly, if y is restricted to any finite set of reals, such an x can be found. Idealization tells us that such an x can be found which is smaller than *any* standard y. This is how infinitesimals make their appearance.

(S) *Standardization.* Let $C(z)$ be any formula with at least the free variable z.

$$\forall^{\text{st}} x \exists^{\text{st}} y \forall^{\text{st}} z (z \in y \Leftrightarrow z \in x \wedge C(z)).$$

This supplements the Axiom Schema of Separation (Suppes 1960) by showing how to form standard sets by imposing conditions on elements of already existing standard sets. It is necessary since the set building of Separation makes no mention of "standard".

(T) *Transfer*. Let $A(x, t_1, \ldots, t_k)$ be an *internal* formula with free variables x, $t_1, \ldots, t_k$ and *no other free variables.* Then

$$\forall^{\text{st}} t_1 \cdots \forall^{\text{st}} t_k (\forall^{\text{st}} x A(x, t_1, \ldots, t_k) \Rightarrow \forall x A(x, t_1 \ldots t_k)).$$

This says that a formula with all but one variable standard can be tested on standard sets to determine its truth on all sets. By taking contrapositives, we can conclude that if a set with certain properties exists, then a *standard* set with those properties exists. This tells us that any uniquely defined set, such as the integers or reals, must be standard.

(Note that the initials of the axioms are the same as those for Nelson's name for his work: Internal Set Theory.)

The two main results of IST used by MM are the following:

THEOREM 1.1. *Let X be a set. Then every element of X is standard if and only if X is a standard finite set.*

THEOREM 1.2. *There is a finite set F such that for all standard y we have $y \in F$.*

(The proof of this startling result is actually quite simple. Consider the statement $\forall^{\text{st fin}} z \exists x \forall y \in z$ (x is a finite set & $y \in x$). This is clearly true since we can take $x = z$. By Idealization $\exists x \forall^{\text{st}} y$ (x is a finite set & $y \in x$); let $F = x$.)

6. INTERNAL SET THEORY AND ZENO'S PARADOXES

MM's argument makes use of the following principle (McLaughlin and Miller 1992, 378):

(E2) The fact that an object is located at a point in space-time cannot be established if the coordinates describing the point are nonstandard real numbers.

Regarding (Z1), MM note that the set of endpoints of the Z-intervals is the collection of points of the form $1 - 2^{-i}$. Since this is an infinite set, Theorem 1.1 tells us that at least one such endpoint is nonstandard. By their principle, a run terminating at this endpoint must be rejected. Thus, the premise of (Z1), namely that *each* Z-run must be performed, has been shown to be false.

This reasoning can be repeated. Thus, MM are able to claim that only a finite number of Z-runs are needed to complete the race course. This argument is made formal by applying Theorem 1.2 to the set of points that the runner must traverse, i.e., those in the unit interval [0, 1]. (MM consider points on the runner's world-line, but this is not necessary for their argument.) Let $S = [0, 1] \cap F$, where F is the set characterized by Theorem 1.2. Then S is a finite set (in the context of IST) which contains at least all the standard reals in [0, 1].

One might think that S is infinite since the function $x \mapsto \frac{1}{2}x$ appears to be a bijection of S with a proper subset of itself. However, this is not the case because it is impossible to prove that the image of this function is a subset of S. If x is a nonstandard element of S, then there is no guarantee that the nonstandard number $\frac{1}{2}x$ lies in S since F is only guaranteed to contain all *standard* objects. This argument illustrates the nonintuitive nature of IST and the care which must be taken in its use.

Since S is finite, the usual arguments of set theory show that there is a natural number n, and a bijection: $S \leftrightarrow \{z \in \mathbf{N} | z < n\}$. The latter set is a nonstandard proper subset of **N** (the natural numbers), and n is a nonstandard natural number. Denoting by r_i the image in S of the natural number i under this bijection, we obtain $0 = r_0 < r_1 < \cdots < r_n = 1$, a "finite" partition of [0, 1]. Thus, MM have replaced Zeno's infinite set of rational endpoints $1 - 2^{-i}$ with a "finite" set of endpoints. Despite its "finiteness", this partition is exceedingly fine. According to Theorem 1.2, it contains at least all the standard reals as partition points! MM argue that this "finite" partition not only allows a theory of motion replacing SMT, but also resolves the Arrow paradox by virtue of the fact that the arrow's motion is frozen only at the standard reals constituting the partition. The arrow is free to move on the presumably much larger set $[0, 1] - S$.

There are two primary objections to MM's use of IST: first, the use of the word "finite" in the context of nonstandard analysis and, second, the very applicability of nonstandard analysis to the resolution of Zeno's paradoxes.

In his review of Nelson's paper, Davis (1983) points out that Nelson's strict syntactical (formal) view of IST "leads him to use terminology in a way that may appear peculiar" (Davis 1983, 1203). Referring to Nelson's use of the word "finite", he notes that "some may wish to read 'hyperfinite' for 'finite' and 'hyperreal' for 'real' ". The use of *hyperfinite* and *hyperreal* has been widespread by practitioners of nonstandard analysis to indicate that they are using nonstandard models for the reals and that the words *finite* and *real* would be misleading in this context. In referring to the set F which contains all standard reals, Davis writes: "Of course this same result is available in more conventional treatments [of nonstandard analysis] as well. But F then would be called 'hyperfinite' to emphasize that while F shares all the 'internal' properties of finite sets, it is of course infinite when viewed externally [i.e. from outside IST]" (Davis 1983, 1204).

Unfortunately, MM take advantage of this ambiguity. They use the word "finite" in its technical sense, but leave the impression that it retains its intuitive meaning. If they had pointed out that the set S of "checkpoints" which they construct is actually *hyper*finite, as is its cardinality n, their argument would have been far less dramatic and convincing. The natural number n is not an ordinary natural number like 23 or even like a googolplex $(10^{10^{100}})$.

The nonstandard natural number n can never be constructed because, as Nelson proves, anything that can be explicitly constructed using classical methods is a standard object. In some mystical sense, if it were possible to figure out *what* n is, then it could not be *that*. The primary use of IST is to provide elegant alternative proofs of theorems via nonstandard analysis. However, IST also provides a delightful game in which formal definitions, originally based on intuition, can be viewed in new and unfamiliar contexts in which their meaning may be very different from their intuitive one.

Moreover, contrary to what MM state, there is no particular reason to believe that there is any difference in "size" between the set [0, 1] and the set S which they construct. Although S is internally finite, it is nonstandard. It contains *at least* all the standard reals in [0,1], and conceivably many more numbers. In fact, S is not even uniquely defined because F is not uniquely defined (Nelson 1977).

Concerning the nature of IST and nonstandard analysis in general, Nelson quotes Robinson's comment that "from a formalist point of view we may look at our theory syntactically and may consider that what we have done is to introduce *new deductive procedures* rather than new mathematical entities" (Nelson 1977, 1198). It should be noted that nonstandard analysis was initially created not to introduce new deductive procedures, but to justify *old* deductive procedures based on the infinitesimals of Newton and Leibnitz. Bishop Berkeley (1734) dismissed infinitesimals as "ghosts of vanished quantities". Whether nonstandard analysis has resurrected these ghosts is debatable, but the work of Robinson, Nelson, and others has shown that the "new deductive procedures" based on infinitesimals can provide a powerful and legal mnemonic for the more complicated arguments using limits. However, there is a price to be paid for the powerful formalism of non-standard analysis, and IST in particular, namely the extremely nonintuitive properties of some internally finite sets and the philosophical problems raised by the non-experiential nature of infinitesimals.

From the point of view of mathematical formalism, there is no objection to numbers that are internally finite but externally infinite. Even intuitionists allow the formalists their games. If MM were trying to explain a logical paradox using formalism then there would be no problem. But Zeno's paradoxes are different; they are not based on an argument of logic, but rather upon our supposed intuitive rejection of infinite processes. Infinite processes can be accommodated by formal mathematical models. Nearly 70 years ago Russell showed that infinite sets, sums, cardinals and ordinals provide a sound basis for analyzing motion (Russell 1929). And modern measure theory, utilizing limiting operations over infinite sets, shows how sets of positive measure can be comprised of points of measure 0 (Rudin 1966).

It is not mathematics which stands in the way of the infinite, but rather the intuition that anything which requires the performance of an infinite number of acts is inherently unintelligible, if not self-contradictory. If MM reject SMT because it fails to address this intuitive issue, then they should not accept IST simply because it uses the word "finite" instead of "hyperfinite" or "internally finite". In their concluding paragraph, they write: “It should be noted that some of the sets utilized in the present work can be viewed ‘externally’ (Nelson 1977) wherein they have infinite cardinality. However, this fact does not affect the validity of the treatment of Zeno’s objections within an IST-modeled universe.” In isolation this comment is formally true, but MM go further and state that the use of IST is to be “interpreted within an empirical context” (McLaughlin and Miller, 1992, 371). Given the counterintuitive meaning of “finite” in IST, why should one choose to view the universe through the lens of IST? Indeed, IST, which applies the predicate “finite” to a set that contains all the standard reals, does not relieve the uneasiness caused by Zeno’s paradoxes. It only explains the obscure by the more obscure. In summary, Internal Set Theory is not useful in analyses of Zeno’s paradoxes of motion because its application requires a model of the world that is not in accordance with experience or intuition.

7. TIME AND MOTION IN ZENO’S PARADOXES

Vlastos has suggested that it is a category-mistake to refer to motion occurring at an instant of time (Vlastos 1967, 373). In analyzing motion, time can only be discussed in terms of intervals or duration. In fact, we would go further and reverse the priority of the concepts of time and motion. Motion should be considered to be the fundamental concept. While humans seem to have a qualitative sense of the passage of time, the quantitative concept of time is derived from the concepts of space and motion. If nothing in the world changed, there would be no need for the measurement of time and, moreover, no possible definition of time. The measurement of time was invented to describe change.

A motion is a change, perceived by some observer, in an object’s position with respect to other objects considered by that observer to be stationary. Certain types of motions are periodic; the moving object “periodically” returns to its initial position. Humans have an inborn ability to sense such periodicity. A unit of time is defined in terms of the period of some motion that is subjectively judged to be uniform, such as the length of a day, a breath, or a heartbeat. (Galileo measured the period of a pendulum by counting his pulse.) Refinements in the measurement of time are made by the discoveries of other processes that appear to be more uniform, e.g., the vibration of a quartz crystal or the frequency of some atomic transition.

Once it is realized that motion and not time is fundamental, Zeno’s paradox of the Arrow dissolves. One should not ask whether the arrow is moving *at* a particular time because time is a characteristic of motion, motion is change, and there is *no* change at a point. It is only when we are trapped by the verbal snare of speaking of motion at a point in time that we get confused. No one has ever perceived motion stopped at a point. No matter how fast the shutter or quick the eye, there is still a finite interval of change, still a slight blur.

This position is not inconsistent with the use of derivatives to define the velocity at an instant of time. Velocity, defined as a derivative, is a limit of measurements of position and time defined throughout an interval. Thus, the derivative does not define motion at a point, it merely assigns to a point a number which depends on motion already defined in a neighborhood of the point. As many beginning calculus students have been told: "dy/dx is not a quotient, it is the *limit* of quotients."

We may *perceive* a motion either all at once, or made of several pieces. We never perceive a motion as an infinite number of pieces since this would entail perceiving infinitely small changes. Although we can perceive a given motion only once (we can never step in the same stream twice), we can *conceive* of the same motion many different ways. Vlastos' distinction between Z_a- and Z_b-runs arises in this manner. Consider again the run from 0 to 1 traversed at constant velocity. The Z_a-run is the run perceived as a single uninterrupted motion. The Z_b-runs are the various conceptual subdivisions of the perceptual run. Let us call the conception of the run that coincides with the perception of the uninterrupted motion the *canonical run*. The various other possible conceptual runs are all partitions of the unit interval and can all be constructed from the canonical run. Thus, [0, $\frac{1}{2}$, 1] and [0, $\frac{1}{2}$, $\frac{3}{4}$] are conceptual runs. Zeno's partition is the conceptual run [0, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\ldots$]. Since all such conceptual runs are derived from the same canonical run, they must satisfy a certain consistency condition to be discussed in the following section.

On a perceptual level there are no paradoxes since no one can perceive infinitely small runs at either the beginning, end, or middle of the race course. In (Z1), the runner reaches the end because he reaches the limit of our perception of small distances and small changes and simply slips over the last intervening distance without us observing him do so. A similar argument applies to (Z2).

On the conceptual level there is complete freedom of choice. Conceptual runs exist solely in the mind and, in fact, are subject only to those conditions that a particular mind chooses to impose on them. Those who believe they can conceive of infinity (like *Inf* in the dialogue below) are perfectly free to choose a series of runs with infinitely many pieces; they simply reject Zeno's claim that such actions are impossible. Those who do not believe they can conceive of infinitely many acts or runs (like *Fin*) will simply not do so. Now let us listen in on their conversation.

Inf You must concede that Achilles has to pass through every point $1 - 2^{-n}$.

Fin Yes I do. Name any n and I'll conceive of the run $[0, \frac{1}{2}, \frac{3}{4}, \ldots, 1-2^{-n}]$.

Inf But you haven't reached 1.

Fin $[0, \frac{1}{2}, \frac{3}{4}, \ldots, 1 - 2^{-n}, 1]$: no problema!

Inf But what about $1 - 2^{-(n+1)}$?

Fin You never asked about that. But I certainly have included it, it's just not an endpoint. However, I'll throw it in explicitly by adding an extra interval if that will make you happy.

Inf You must throw them *all* in.

Fin But why, if I can conceive of a run that consists of any finite collection of them which you can name?

Inf Because every one of them must be undertaken.

Fin "Undertaken" is a misleading word; it sounds as if a special effort or delineation must be made for each interval traversed. In fact, I am incapable of conceiving of an infinite number of undertakings, since I have never perceived such a thing in my life. I just conceive of doing more and more Z-runs until they get so small I can't conceive of them, at which point I just move Achilles the last tiny bit to the end of the racetrack, by throwing in a 1 and closing the path. This is exactly consistent with my perceptions and my intuition. By the way, how do *you* reach 1, since none of your Z-runs contain 1?

Inf That's part of my conception of infinity: when I have imagined these infinitely many Z-runs having taken place, I am indeed at 1.

Fin Aha! I get it. You're at one with the infinite; I'm at one with the finite.

How does *Inf*, or anyone, conceive of the infinite collection of Z-runs? Possibly by some cosmic insight, but more likely with the aid of that great facilitator of conceptions, mathematics.

8. RATIONAL INTERVALS AND THE REAL NUMBERS

The real numbers can be constructed as families of rational intervals, and their algebraic properties derived from "interval arithmetic". This approach to the reals, based on the fundamental nature of scientific measurement, is due to G. Stolzenberg (1990). It is more suitable for dealing with the interval-like nature of time and motion than the traditional approaches of constructing the real numbers using Dedekind cuts or Cauchy sequences.

We begin by noting that once units have been chosen for distance and time, all measurements involve *rational* multiples of these units. Furthermore, measurements are never exact; a measured value is actually an interval of rational numbers giving a range of possible measurements of an observed quantity. To make these ideas precise, for any rationals $r \leq s$, the symbol $[r, s]$ is used to denote the set of rationals x such that $r \leq x \leq s$. We call this set a *rational interval*. Its *length* is $s - r$.

If two measurements are made of the same quantity, we expect the rational intervals obtained to overlap. We call this property *consistency*. For example, in measuring the circumference of a circle whose diameter is 1, we might obtain the intervals [3.11, 3.145] and [3.14, 3.15], both of which contain, for example, the rational 3.142. These intervals are consistent. We note that sometimes two measurements of the same quantity do not satisfy the consistency condition. As of early 1995, the physical/chemical theory of stars date the "Big Bang" in the interval $[-18, -14]$ (billions of years ago), whereas the Cepheid variable information from the Hubble telescope place this date in the interval $[-12, -8]$. These intervals are *inconsistent*.

Not only do we expect our measurements to be consistent, but we expect them to become more precise as measurement devices improve. A more precise measurement is reflected in a smaller interval.

The fundamental definitions and theorem characterizing the real numbers are:

DEFINITION 1. A real number is a family of rational intervals with the following two properties.

(Consistency): Any two intervals have a non-empty intersection.

(Fineness): For any rational $\epsilon > 0$, there is an interval of length less than ϵ.

A rational number r has a particularly simple representation as the family consisting of the single interval $[r, r]$.

One can define the usual algebraic operations on real numbers by defining them on rational intervals and hence on families of rational intervals. For example, $[r, s] + [u, v] = \{x + y | \, r \leq x \leq s \,\&\, u \leq y \leq v\} = [r+u, s+v]$. We will omit further details of this interval arithmetic. However, we do need to define order relations on reals.

DEFINITION 2. Two real numbers A and B are equal if every interval of A meets every interval of B.

DEFINITION 3. $A < B$ if there is *some* interval of A which lies wholly to the left of some interval of B. We write $A \leq B$ if it is not true that $B < A$.

DEFINITION 4. The real interval $[A, B]$ consists of the reals X such that $A \leq X \leq B$.

THEOREM (Completeness). Let F be a family of real intervals which satisfy the consistency and fineness conditions. Then there is a unique real number X which belongs to every one of the intervals. X is called the *limit* of the family F.

We now apply these ideas to time and space, which are to be modeled by the real numbers. In identifying a point in time, we need only the assumption that consistent and arbitrarily fine, but not infinitely fine, measurements can be performed. Given any $\epsilon > 0$, we need only know how to construct a measurement of tolerance less than ϵ. It is not necessary to perform the impossible task of measuring any time exactly. A point in time, as a real number, is determined by a family of consistent and arbitrarily fine time intervals.

For the purposes of analyzing Zeno's paradoxes, we define a *motion* to be a continuous function ϕ: $I \to R$, where I is a closed finite subinterval of R, the reals. Mathematically, such a function is generally defined on the reals by first specifying it on the rationals and then checking that $|\phi(x) - \phi(y)|$ can be made small if x, $y \in [r, s]$ and the length of $[r, s]$ is made sufficiently small (uniform continuity on finite intervals). Letting $[r, s]$ range over a family defining the real A, we can then define $\phi(A)$ using the completeness theorem. This process is sometimes called "extending by continuity". Any motion that could actually be observed and measured would have to be defined in this manner. In addition, any theoretical motion described by algebraic or analytical methods could also be defined in this way, since the algebraic or analytical functions of mathematics can be defined via rational arithmetic and extension by continuity. Such a program is often carried out (or at least described) in a modern analysis course using, for example, Cauchy sequences or Dedekind cuts instead of interval arithmetic.

Now we resolve the Arrow paradox using this formalism. To say that the arrow is at point P at time t (both real numbers) is to say that $P = \phi(t)$. In the light of the previous discussion, this means that ϕ is defined on the collection of rational intervals that define t, and that P is determined by the behavior of ϕ on these intervals. There is no motion *at* a point, just motions in arbitrarily fine neighborhoods of a point. The position P is defined by these motions. We have thus provided an exact mathematical description of position at an instant of time in terms of motion in an interval of time. In a similar way we could also define the derivative of ϕ at t. This provides us with a measure of velocity at an instant of time, also in terms of motion in an interval.

Using the definition of motion as a function $\phi : I \to R$, we can decompose any motion into submotions or subruns. Decompose the interval I into a family of subintervals $\{J_\alpha\}_{\alpha=1,2,\ldots}$ so that $\cup_\alpha J_\alpha = I$. Denote by ϕ_α the function $\phi|J_\alpha$ (ϕ restricted to J_α); then $\phi_a : J_\alpha \to R$ for each α. These functions satisfy, for each pair α, α', the following consistency condition:

$$\phi_\alpha|J_\alpha \cap J_{\alpha'} = \phi_{\alpha'}|J_\alpha \cap J_{\alpha'}.$$

This is a description of the motion or run ϕ, in the Z_b sense, as a sequence of subruns ϕ_α. Furthermore, if $I = \cup_\beta K_\beta$ is a different decomposition of I into subintervals, then the functions $\phi_\beta = \phi|K_\beta$ provide us with a different description of the same motion ϕ.

Now consider the converse situation. Instead of beginning with ϕ, start with any particular decomposition of I into subintervals, $I = \cup_\alpha J_\alpha$, and motions $\phi_\alpha : J_\alpha \to R$ on each of the subintervals. If the consistency condition is satisfied for each pair ϕ_α, $\phi_{\alpha'}$, then together these individual motions define a (unique) motion ϕ on all of I, whose restriction to each J_α is precisely ϕ_α. Thus, a motion on a whole (time) interval is exactly equivalent to a consistent family of motions on a family of subintervals. There are as many different descriptions of a motion $\phi : I \to R$ as there are subdivisions of I. Using this description we can clarify both *Fin*'s and *Inf*'s arguments in mathematical terms. *Fin*'s denial of the necessity of conceiving the total run as the collection of Z-runs is simply the assertion that any description of a motion in terms of a consistent family of submotions is as good as any other such description. No one description should be considered to be the "correct one". Thus, *Fin* chooses to use a finite union of intervals J_α corresponding to his finite perception of the motion. *Inf* uses an infinite union of intervals K_β corresponding to her Zenonian description of the motion.

Now we defend *Inf*'s claim that the runner is *at* 1 after all the Z-runs, even though no Z-run terminates with 1. Note first that

$$\bigcup_{k=1}^{\infty} Z_k = [0, 1)$$

where $Z_k = [1 - 2^{-(k-1)}, 1 - 2^{-k}]$.

This interval [0, 1) with 1 omitted seems to put *Inf* on the spot, but this is a misinterpretation. The infinite union does not represent the result of infinitely many Z-runs! Rather, it represents the result of running any finite number of them. This surprising fact is perhaps more easily understood by writing

$$\bigcup_{k=1}^{\infty} Z_k = \bigcup_{k=1}^{\infty} \left(\bigcup_{i=1}^{k} Z_i \right).$$

A point is in a union of sets if it is in (at least) one of them. Thus, a point is in the infinite union on the right of this equation if it is in one of the *finite* unions; in other words, it is in the infinite union if it is part of some finite Z-run. The half-open interval [0, 1) therefore just represents the results of *finite* Z-runs. To see where the runner is after *all* possible Z-runs, we must define this position as a real number, i.e. as a family of rational intervals. After the kth Z-run, he is at the point $1 - 2^{-k}$ which is in the rational interval $[1 - 2^{-k}, 1]$. Thus, we can take as our family the collection of all such rational intervals $[1 - 2^{-k}, 1]$, $k = 1, 2, \ldots$ This family clearly contains intervals of arbitrarily small length, and 1 is in each interval. By Definition 2, the real number defined by this family is precisely 1 (represented as [1, 1]). Thus, *Inf* is perfectly correct in conceiving the result of the infinite process of performing all Z-runs as reaching 1.

The point of this argument has been to construct the real number representing the runner's final position. It is crucial to note that no limit is involved. In the construction of a real number based on the completeness theorem, the data is a family of *real* intervals that is consistent and has intervals of arbitrarily small length. The limit of such a family is the unique real number that lies in each interval of the family. The situation for the runner is different. Here the data for his final position consists of a family of *rational* intervals that satisfies consistency and fineness. *A priori*, there is no reason to suspect that such a family has a unique common element. We do not take a limit of such a family because there is no need to. By the definition given above, such a family in itself *is* the real number. In addition, we have shown that this particular number, defined by a family of rational intervals, equals the real number 1, defined by the family consisting of the single interval [1, 1]. That these two families both represent the same number follows from the definition of equality: the interval [1, 1] meets each rational interval $[1 - 2^{-k}, 1]$.

The definition of a real number using rational intervals is consistent with the constructivist philosophy of mathematics. The family defining a real number may be as simple as a single interval $[r, r]$, defining the rational number r as a real, or as complicated as a family defined by partial sums of a convergent series. Even though the family may be infinite, constructive procedures may be used to determine whether the consistency and fineness conditions are satisfied and whether two reals are equal or unequal.

We conclude our presentation of mathematical approaches to Zeno's paradoxes by noting that, unlike models based on IST, the model based on families of rational intervals provides a realistic interpretation of physical measurements and their role in the representation of motion.

9. THE PHYSICS OF ZENO'S PARADOXES

Vlastos' distinction between Z_a- and Z_b-runs was a major contribution in understanding the problem of completing the infinite number of tasks involved in Zeno's Dichotomy and Achilles paradoxes. However, Vlastos never explains why in discussing physical motion it is inappropriate to use the idea of Z_b-runs. We believe that the distinction between Z_a-runs and Z_b-runs captures the distinction between the physics that is involved in the motion of any real object and a purely mathematical model of the relationship between distance and time, involving such concepts as intervals, subintervals, and infinite sums of subintervals.

Mathematically, there is no problem with defining the sum of the infinite series used in the analysis of the Dichotomy. The infinite number of tasks can be completed because the sum of the infinite series $\sum(1/2^n)$ is well-defined, and there is no need for the distinction between Z_a- and Z_b-runs. The two types of runs are alternative, equally valid descriptions of the same motion on the unit interval $0 \leq t \leq 1$.

However, for the actual motion of a real object, Z_a-runs are distinguishable from Z_b-runs. In order to decompose a run into successive Z_a-traversals of subintervals, it is necessary that the completion of each sub-interval be indicated in some manner. Either some sign must be given by the moving object at the end of each subinterval indicating that it has completed that subinterval or some measuring device interacting with the moving object must emit a signal marking the completion of each subinterval. In his analysis of the Race Course, Grünbaum gives several examples of runs that involve signs produced by the runner (Grünbaum 1968, 82–86). In the two versions of the staccato run described above in Section 4, each subinterval is marked by a pause in the motion. In another example, also proposed by Grünbaum, the runner plants a flag on completing each subinterval. For an example of a signal, consider a set of photo-emission devices, one stationed at the end of each subinterval. Each device transmits a light beam to a detector as the object completes its subinterval.

Since Zeno's paradoxes are about the motion of physical objects, we now ask whether the implementation of the infinite set of the signs or signals required to indicate the completion of each of the subintervals violates any of the laws of physics. From a practical point of view, it is of course impossible to either generate or process this infinite collection of signs or signals. However, in this discussion, as in those of Grünbaum (1968), Thomson (1970a and 1970b), and Benacerraf (1970) previously cited, we do not consider practical limitations. We assume that a measurement of time or position can be made with arbitrary precision and accuracy, as described in the previous section. Moreover, we ignore questions concerning the confirmation of the traversal of an interval that may be shorter than the diameter of an atom. Nor are we concerned that an infinite number of signalling devices, such as photo-emission devices, would be required to monitor all of the subintervals or that there is no room for all of these detectors.

The fundamental difficulty with implementing the signs and signals required to indicate the completion of each of the subintervals involves the Heisenberg uncertainty principle. (See Gottfried 1966 for details of quantum theory.) This principle applies to all signs and signals regardless of the details of the signing or signalling device. In order to confirm the completion of each subinterval at the appropriate time, the runner's position must be known to an increasingly high degree of precision as he approaches the end of the entire interval. One form of the uncertainty principle states that if the uncertainty in position of some material object becomes small as the result of a precise measurement of its position, then the uncertainty in its momentum, and hence in its velocity, becomes large.

Suppose we make a determination of the position of the runner at the time we expect the runner to complete the nth subinterval. The uncertainty in this measurement must be small enough so that we can be sure that the runner has indeed completed the subinterval at the appropriate time. According to the uncertainty principle, as the uncertainty in our knowledge of the runner's position is reduced, the uncertainty in the runner's velocity must increase. As a result, it becomes less likely that a measurement of the runner's position at the time we expect the runner to complete the $(n+1)$st subinterval will show that the runner is indeed completing that subinterval. As n becomes large, the likelihood that the runner will be observed in the appropriate subinterval becomes minuscule. The descriptions of the Race Course in terms of Z_a- and Z_b-runs are no longer equivalent.

It is important to note that this argument is completely general. In particular, it applies to the staccato run and its modifications, to the pole plant example and any conceivable refinements of it. Once the motion of an object is interfered with as the result of a measurement of position, its trajectory changes in an uncontrollable fashion. Unlike the situation in classical mechanics, according to quantum mechanics it is impossible to continually refine an experimental set-up so that the effects of the disturbances to the system caused by a measurement can be reduced to a level that is arbitrarily small.

It is not difficult to illustrate the application of the uncertainty relation to the analysis of the Race Course. Denote the uncertainty in the position and in the momentum of the runner by Δx and Δp, respectively. The uncertainty relationship states that

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}.$$

In this equation, $\hbar$, Planck's constant divided by 2π, is equal to 1.0546×10^{-34} Joule-sec.

Suppose the racetrack has length 1 m (meter) and is traversed in 1 s (second). These values are chosen for convenience. The velocity of the runner is 1 m/s; the nth subinterval has length 2^{-n} and it is traversed in 2^{-n} seconds. Assume, further, that the mass of the runner is 1 kg (kilogram). Consider the 150th subinterval. This interval has length 2^{-150} m, which is equal to 7.01×10^{-46} m. Assume that the measurement of position as the runner approaches the end of this subinterval has a relative uncertainty of 10%. Then Δx equals 7.01×10^{-47} m. The uncertainty in the momentum, Δp, is then 7.52×10^{11} kg m/s. The uncertainty in the velocity, Δv, is the value of the uncertainty in momentum divided by the mass of the runner. and so is numerically equal to the uncertainty in momentum. Thus, the value of the uncertainty in the velocity is a factor of 752 billion times the value of the velocity itself. By the time the runner is supposed to complete the 151st subinterval, we will have no idea of where the runner is. Note that changing the length of the race course, the time of traversal, or the mass of the runner, even by factors of one hundred or one thousand, has little effect on this argument. We might also point out that the length of the 150th interval is approximately 36 orders of magnitude smaller than the size of an atom.

10. SUMMARY

MM applied Internal Set Theory to Zeno's paradoxes of motion in the hopes of avoiding the use of infinities that Zeno's description seems to entail. Although Internal Set Theory is useful as a formal tool for proving mathematical theorems, its terminology and many of the results deducible from it are not consistent with our intuitive notions of finiteness and our perception of the real world. We therefore reject it as a possible technique for analyzing Zeno's paradoxes.

In place of IST we substitute a constructive definition of the real number continuum, in which a real number is a family of intervals of rational numbers (representing measurements) satisfying conditions of consistency and fineness. This definition gives us a tool with which we are intuitively comfortable and which represents the actual nature of scientific measurement. The result of any scientific measurement is a rational interval, e.g., 2.859 ± 0.004, and every real number is also a family of rational intervals. Thus the questionable use of limits to capture the idea that the "true" value of some measured quantity is a real number is replaced by the verification that increasingly precise measurements of some quantity are consistent and can, in principle, be made arbitrarily fine.

The notion of a real number as a family of intervals is particularly felicitous when analyzing motion and time. Since periodic motion is used to measure time, we can only speak of time quantitatively in terms of intervals. The classical definition of a motion as a function $\phi : I \to R$ is then interpreted by considering a point τ of the domain I as a family of rational intervals (measurements of time). Via ϕ, these intervals determine a corresponding family of intervals (measurements of position) in R. The value $\phi(\tau)$ is then the real number determined by this family of position intervals. Thus, position at a "point in time" is defined in terms of motion on an interval. Using derivatives, velocity can be defined in a similar fashion. This approach leads to a natural resolution of the Arrow paradox.

In order to resolve the Dichotomy, we combine this model of motion with Vlastos' distinction between Z_a and Z_b-runs. A motion through an interval I can be described in many ways as motions on families of subintervals of I. Thus, in a conceptual sense, motion through some distance may be described equally effectively as a single motion or as a collection of possibly infinitely many submotions.

Finally, in a perceptual sense, we enquire whether physical measurements can detect infinitely many submotions in a finite interval of time. An affirmative answer requires that an infinite number of discrete acts be performed in a finite time, since each measurement entails delineating a submotion by some physical event. An arbitrarily precise infinite set of measurements can be made without contradicting classical physics, but can not be made in the context of quantum mechanics, due to the Heisenberg uncertainty principle.

On the conceptual level, quantum mechanical restrictions do not apply. As shown above in the discussion of the use of intervals in interpreting motion, we are free to accept or reject the demand that the Race Course be run as an infinite sequence of decreasing intervals. Either choice can be made in formulating an intuitively and mathematically satisfying resolution of Zeno's paradoxes of motion.

NOTE

* We would like to thank Professor Gabriel Stolzenberg for the use of his lecture notes on real analysis and for many helpful discussions.

REFERENCES

Benacerraf, P.: 1970, 'Tasks, Super-Tasks, and the Modern Eleatics', in W. C. Salmon (ed.), *Zeno's Paradoxes*, Bobbs-Merrill, Indianapolis and New York, pp. 103–129.
Berkeley, G.: 1734, *The Analyst*, excerpted in J. R. Newman (ed.), 1956 *The World of Mathematics*, Simon and Schuster, New York, vol. 1, pp. 288–293.
Davis, M.: 1983, Review of E. Nelson's 'Internal Set Theory: A New Approach to Non-standard Analysis', *Journal of Symbolic Logic* **48**, 1203–204.
Gottfried, K.: 1966, *Quantum Mechanics*, W. A. Benjamin, New York.
Grünbaum, A.: 1968, *Modern Science and Zeno's Paradoxes*, George Allen and Unwin Ltd, London.
Grünbaum, A.: 1970, 'Modern Science and Zeno's Paradoxes of Motion', in W. C. Salmon (ed.), *Zeno's Paradoxes*, Bobbs-Merrill, Indianapolis and New York, pp. 200–250.
James, W.: 1948, *Some Problems of Philosophy*, Longmans, Green and Co., Ltd., London.
McLaughlin, W. I.: 1994, 'Resolving Zeno's Paradoxes', *Scientific American*, November, 84–89.
McLaughlin, W. I. and Miller, S. L.: 1992, 'An Epistemological Use of Non-standard Analysis to Answer Zeno's Objections Against Motion', *Synthese* **92**, 371–84.
Nelson, E.: 1977, 'Internal Set Theory: A New Approach to Nonstandard Analysis', *Bulletin of the American Mathematical Society* **83**, 1165–98.
Robinson, A.: 1966, *Non-standard Analysis*, North Holland, Amsterdam. (2nd ed., American Elsevier, New York, 1974).
Rudin, W.: 1966, *Real and Complex Analysis*, McGraw-Hill, New York.
Russell, B.: 1929, *Our Knowledge of the External World*, W. W. Norton, New York.
Salmon, W. C.: 1970, 'Introduction' in W. C. Salmon (ed.), *Zeno's Paradoxes*, Bobbs-Merrill, Indianapolis and New York, pp. 5–44.
Smart, J. J. C.: 1967, 'Time', in P. Edwards (ed.), *The Encyclopedia of Philosophy*, Macmillan, New York, Vol. 8, pp. 126–134.
Steen, L. A.: 1971, 'New Models of the Real-Number Line', *Scientific American*, August, 92–99.
Stolzenberg, G.: 1990, 'Notes on Real Analysis', unpublished.
Suppes, P.: 1960, *Axiomatic Set Theory*, D. Van Nostrand, New Jersey.
Thomson, J.: 1970a, 'Tasks and Super-Tasks', in W. C. Salmon (ed.), *Zeno's Paradoxes*, Bobbs-Merrill, Indianapolis and New York, pp. 89–102.
Thomson, J.: 1970b, 'Comments on Professor Benacerraf's Paper', in W. C. Salmon (ed.), *Zeno's Paradoxes*, Bobbs-Merrill, Indianapolis and New York, pp. 130–138.
Vlastos, G.: 1967, 'Zeno of Elea', in P. Edwards (ed.), *The Encyclopedia of Philosophy*, Macmillan, New York, Vol. 8, pp. 369–79.

Whitehead, A. N.: 1948, *Process and Reality*, Macmillan, New York.

Joseph S. Alper
Department of Chemistry
University of Massachusetts – Boston
Boston, MA 02125

Mark Bridger
Department of Mathematics
Northeastern University
Boston, MA 02115