

THE MATHEMATICS OF PHYSICAL QUANTITIES

PART I: MATHEMATICAL MODELS FOR MEASUREMENT

HASSLER WHITNEY, Institute for Advanced Study

INTRODUCTION

1. Purpose of this paper. To set up a physical theory, one constructs a mathematical model, and considers its relation to certain aspects of the physical world. There is a variety of models associated with the concept of measurement. Certain systems of numbers are important; for instance:

N , the natural numbers;	R^+ , the positive reals;
Q^+ , the positive rationals;	R , the reals.

Commonly one takes R^+ or R as a model for measurement. There are disadvantages in this, however. These models contain a specific number 1, and there is no natural way of putting this number in correspondence with a particular measurement; moreover, the models contain an operation of multiplication, with no natural physical counterpart.

Let us consider the problem of choosing a model M for masses. An object A has a certain property which we call its "mass"; why not let this property itself be an element of the model? As far as the structure of the model is concerned, we need not theorize on what "mass" really is; we need merely give it certain properties in the model. For instance, if we have distinct objects A and B , with masses m_A and m_B , we may think of the objects as forming a single object C ; its mass m_C should then be $m_A + m_B$. Therefore M should contain an operation of addition, and any further properties we choose.

The two types of models that best fit in most situations we shall call "rays" and "birays." A ray (like a half line) is used for positive measurements, and a biray (like an oriented line with starting point), for measurements of quantities both positive and negative. It turns out that numbers appear in a natural way as operators on the model (see the next section).

In this paper we set up the theory of rays and birays; the real number system is constructed along with the models in a natural manner. In fact, this approach gives a simple and elegant way of introducing the reals and finding their basic properties.

We shall not give applications of the models in the body of the paper; some remarks on the subject will be made in this introduction. In this connection, see Part II, which will appear in the next issue of this MONTHLY.

2. Numbers as operators on the models. If we choose a stick of length l , and wish to use it to measure another stick, we lay out the first stick along the second several times; we thus form $l+l$, $l+l+l$, and so on. We also call these lengths $2l$, $3l$, \dots . Thus N appears as a natural set of operators on our model. For masses, there is a different physical process of addition; but again we may use $2m = m + m$ and so on, with the same set N of numbers.

If we have a stick, whose length we call l , and we find a shorter stick, of length l' , such that $3l' = l$, then we wish to give an expression of l' in terms of l . It is natural to set $l' = (1/3)l$. We may now set $2l' = (2/3)l$, and thus introduce Q^+ as operators. Finally, if our model has a certain completeness property, we may enlarge Q^+ to R^+ as operator system, and if we have negative quantities, we may enlarge R^+ to R .

3. Working in the model. Various properties of a model and its operations have obvious meaning in the applications. For instance we have distributive and associative laws:

$$\begin{aligned} 5 \text{ cakes} + 2 \text{ cakes} &= (5 + 2) \text{ cakes} = 7 \text{ cakes} \\ 2 \text{ yd} &= 2(3 \text{ ft}) = (2 \times 3) \text{ ft} = 6 \text{ ft}. \end{aligned}$$

The fact that "2 yd" and "6 ft" name the same element of the model enables us to say they are equal; there is no need for such mysterious phrases as "2 yd measures the same as 6 ft."

4. The use of units. If we wish to use R^+ as a model for measuring (positive) lengths, we must decide which length 1 corresponds to; this length will then be called our "unit length." The more natural model is a ray L (on which R^+ operates); since the elements of L themselves are "lengths," the above question does not arise.

If we choose a length $l_0 \in L$, and compare other lengths with it, we may call l_0 our "unit"; this serves merely to remind us that l_0 is being kept fixed for a period. Suppose we now find certain other lengths, for instance $5l_0, 2l_0, 7l_0$. If we wish to shorten our notations, and call these lengths 5, 2, 7, we are then replacing L by R^+ . We can then say "the length 5 really means the length $5l_0$." More awkwardly, one could say "the length is 5 when measured in terms of l_0 ."

Suppose we wish to "change units," say from ft to in. Then since, for any $a \in R^+$, $a \text{ ft} = a(12 \text{ in}) = 12a$ in, we would replace "the length a " by "the length $12a$." If any problems about units arise, they are at once resolved by going back to the explicit phrase " $a \text{ ft}$."

5. The postulational treatment. Though all rays (and all birays) have the same structure, one may wish to use several rays in a single investigation. For instance, in mechanics, one uses separate rays M, L, T for measurement of mass, length and time. (We study structures containing several rays in Part II.) Hence we introduce our models postulationally; the definitions show whether or not a given structure is a ray or a biray. However, only a single structure R (or one of its subsets) is needed for operators; hence we introduce R constructively. (We give the characterization of R as a complete ordered field at the end.)

A basic theorem in the subject is an *isomorphism theorem*; a homomorphism of one ray into another is necessarily an isomorphism onto, and has certain additional properties (and similarly for birays). This theorem is a great aid in setting up the theory; in particular, with its use, multiplication in R^+ and in R is introduced and its properties derived with a minimal effort.

The postulates used for rays and birays are few in number and simple in character, and correspond to simple experimental phenomena.

6. Other models. We introduce only the most important models. With the real numbers at our disposal, and the facts about rays and birays, other similar models are easily studied. For instance, in measuring masses, one wishes to allow the mass zero (not present in a ray). This extra element may be introduced and related to the remaining elements in the obvious manner.

A model of a somewhat different nature is an oriented affine one-dimensional space T^* ; this is the natural model for instance for moments in time (or positions on a line). There is a corresponding biray T of translations of T^* ; this is the natural model for intervals of time (or directed lengths). We do not consider models including for instance 3-dimensional space; the term "measurement" is not the best term here.

If the measures of some type of quantity form a progression, as in counting, it is natural to use N for a model. However, if several such types of quantities are considered together, it is better to use several isomorphic models. For a plebeian illustration, suppose there will be six children at a party. We wish each to have two balloons and three cookies. What is the total supply needed? The answer is:

$$6(2 \text{ bl} + 3 \text{ ck}) = 6(2 \text{ bl}) + 6(3 \text{ ck}) = 12 \text{ bl} + 18 \text{ ck}.$$

7. A finite model. We have no infinite sets available in our environment. What happens if we have a set G with a large number n of elements, called $1', 2', \dots, n'$, and wish it to approximate to N ? We could define $a' + b'$ to be $(a + b)'$, or n' if $n < a + b$. Note that we may set $a'b' = ab'$; now G operates on itself, thus defining multiplication in G . We find that these operations are commutative and associative, and the distributive laws hold. However, the cancellation laws fail.

We give an instance from everyday life. Helen is setting the table for lunch for four; she places two spoons at each place. Mother answers the doorbell; it is Mr. and Mrs. Jones. Perhaps they will stay; Helen needs 4 more spoons. There are only two left in the drawer, so Helen puts them out. (She thus makes $8_s + 4_s = 10_s$.) Hearing the visitors say goodbye, Helen thinks, take away four spoons. She then realizes that, actually, she must take away only two. In her model, $8_s + 4_s = 8_s + 2_s$.

CHAPTER I. DIVISIBLE SEMI-GROUPS

This chapter is preliminary in nature; certain basic properties of rather general structures are derived. It is shown how N appears as a system of operators on any commutative semi-group, and Q^+ , if the semi-group is "uniquely divisible." To save space and help in the grasping of concepts, the proofs pertaining to N are given in rather intuitive fashion. However, the Peano postulates are seen to hold for N ; hence one may replace the proofs by the usual more formal proofs where desired.

8. Commutative semi-groups. We begin with the definition. •

DEFINITION 8A. A semi-group $(G, +)$ is a nonvoid set G and an associative binary operation $+$ in G . The semi-group is commutative if addition is commutative. Only the commutative case will be considered here.

We may define $x+y+z$ to mean either $(x+y)+z$ or $x+(y+z)$, since these are the same. More generally, as is well known and easy to see, in any sum, the terms may be written in any order, and parentheses may be inserted or removed at will.

We wish to introduce a shorthand notation for such expressions as $x+x$, $x+x+x$, etc. Since the letter x plays no role here, let us think of it as replaced by a dot. This suggests the expressions

$$(8.1) \quad \begin{array}{ccccccc} x & x+x & x+x+x & x+x+x+x & \cdots \\ (\cdot) & (\cdot \cdot) & (\cdot \cdot \cdot) & (\cdot \cdot \cdot \cdot) & \cdots \end{array}$$

We now consider the new expressions as being names for new objects, forming a set N . It does not matter what these objects are; only their relation to G (or any other semi-group) counts. We also set $1=(\cdot)$, $2=(\cdot\cdot)$, $3=(\cdot\cdot\cdot)$.

We next let N operate on G as follows:

$$(8.2) \quad (\cdot)x = x, \quad (\cdot\cdot)x = x+x, \quad (\cdot\cdot\cdot)x = x+x+x, \text{ etc.}$$

We give an elementary property of the operation:

$$\begin{aligned} (\cdot)(x+y) &= x+y = (\cdot)x + (\cdot)y, \\ (\cdot\cdot)(x+y) &= (x+y) + (x+y) = (x+x) + (y+y) = (\cdot\cdot)x + (\cdot\cdot)y, \\ (\cdot\cdot\cdot)(x+y) &= (x+y) + (x+y) + (x+y) \\ &= (x+x+x) + (y+y+y) = (\cdot\cdot\cdot)x + (\cdot\cdot\cdot)y, \end{aligned}$$

and clearly, in general,

$$(8.3) \quad a(x+y) = ax + ay \quad (a \in N; \quad x, y \in G).$$

With the obvious definition of "successor function" σ in N , the Peano postulates are clear:

- (N₁) The element $(\cdot)=1$ is not a successor: $\sigma x \neq 1$ for all $x \in N$.
- (N₂) For all $x \neq y$, $\sigma x \neq \sigma y$; that is, the function σ is one-one.
- (N₃) For any $N' \subset N$, if $1 \in N'$, and $x \in N'$ implies $\sigma x \in N'$, then $N' = N$.

We could now give proofs involving N in the classical manner, using mathematical induction, defining addition in N , and showing how to give definitions by induction. We shall, however, continue to give intuitive derivations.

If we look at the pattern

$$\begin{array}{ccccccc} (x+x+x) + (x+x) & = & x+x+x+x+x \\ (\cdot \cdot \cdot) & (\cdot \cdot) & (\cdot \cdot \cdot \cdot) & (\cdot \cdot \cdot \cdot \cdot) \end{array}$$

this suggests defining

$$(\cdots) + (\cdots) = (\cdots\cdots).$$

The general rule is clear: To add two elements of N , place the corresponding symbols beside each other, and remove the inner parentheses. The commutative and associative laws are evident; hence $(N, +)$ is a commutative semi-group. Moreover, because of our definition,

$$(\cdots)x + (\cdots)x = (\cdots\cdots)x = [(\cdots) + (\cdots)]x,$$

and more generally,

$$(8.4) \quad (a + b)x = ax + bx \quad (a, b \in N, x \in G).$$

Since $(N, +)$ is a commutative semi-group, we may operate on it by N itself, thus defining *multiplication* in N . This gives, for instance,

$$(\cdots)(\cdots) = (\cdots) + (\cdots), \quad (\cdots)(\cdots) = (\cdots) + (\cdots) + (\cdots).$$

As a consequence,

$$[(\cdots)(\cdots)]x = [(\cdots) + (\cdots)]x = (\cdots)x + (\cdots)x = (\cdots)[(\cdots)x],$$

and more generally,

$$(8.5) \quad (ab)x = a(bx) \quad (a, b \in N, x \in G).$$

Letting N operate on itself, (8.5) and (8.4) give the associative law for multiplication and the distributive law. The commutative law for multiplication must be proved separately. The cancellation law: $x+u=y+u$ implies $x=y$, is obvious from the representation of the elements with dots; a proof with the Peano postulates is easily given.

9. Order in N . If we think of the elements of N as laid out in their natural order, then $x < y$ means that x comes before y . The usual properties of order are clear. For later purposes, we show how to derive the properties from the two following properties of addition in N :

(a) For all $a, b \in N$, $a+b \neq a$.

(b) If $a \neq b$, then either $a+c=b$ or $b+c=a$, for some c .

Now write $a < b$ if $a+c=b$ for some c . The following properties follow at once from the definition:

$$(9.1) \quad a < a + b.$$

$$(9.2) \quad \text{If } a < b \text{ and } b < c \text{ then } a < c.$$

$$(9.3) \quad \text{If } a_i < b_i (i = 1, \cdots, n) \text{ then } \Sigma a_i < \Sigma b_i.$$

The *trichotomy* property is: For all $a, b \in N$, exactly one of the following is true:

$$a < b, \quad a = b, \quad b < a.$$

For if $a \neq b$, then (b) shows that $a < b$ or $b < a$. That at most one of these is true follows from (a). (If $a + c = b$, $b + d = a$, then $a + (c + d) = a$, contradicting (a).)

Order is related to addition by the property (writing "iff" for "if and only if")

$$(9.4) \quad a < b \quad \text{iff} \quad a + c < b + c.$$

This is evident, using the cancellation law. Finally,

$$(9.5) \quad a < b \quad \text{iff} \quad na < nb;$$

for by (9.3), $a < b$, $a = b$, $b < a$ imply respectively $na < nb$, $na = nb$, $nb < na$.

Rather than discuss *subtraction* here, we give the equivalent discussion when studying rays, in section 14.

10. Some examples of semi-groups. We give first a general kind of example:

Example 10A. Let G contain the elements $1, 2, 3, \dots, m, \dots, n$ from N . Let the successor function in G be as in N , except that we set $\sigma n = m$. Now addition is defined in terms of this function; we require

$$(10.1) \quad x + 1 = \sigma x, \quad x + \sigma y = \sigma(x + y).$$

(These relations are used in defining addition in N through the Peano postulates.) For instance, with $m = 5$, $n = 7$, the elements are $1, 2, 3, 4, 5, 6, 7$; adding 3 and adding 4 gives respectively the sequences

$$4, 5, 6, 7, 5, 6, 7; \quad 5, 6, 7, 5, 6, 7, 5.$$

Example 10B. Take $m = 1$ in the last example. Then we have the ring of integers mod n ; n is the zero element.

Example 10C. Take $m = n$ in Example 10A. This gives the example of Section 7.

If G and G' are semi-groups, their *direct sum* consists of all pairs (x, x') with $x \in G$, $x' \in G'$. Define addition componentwise:

$$(x, x') + (y, y') = (x + y, x' + y').$$

This is clearly a semi-group, commutative if G and G' are.

Example 10D. There are ten teaspoons and six dessert spoons in a drawer. This gives us semi-groups G_t and G_d as in section 7. In the direct sum G^* , we have for instance

$$(5_t, 5_d) + (3_t, 3_d) = (8_t, 6_d).$$

If now we do not need to differentiate between teaspoons and dessert spoons, we have sixteen spoons, forming a semi-group G' . There is a natural mapping of G^* into G' , in which for instance

$$(5_t, 5_d) \rightarrow 10_s, \quad (3_t, 3_d) \rightarrow 6_s, \quad (8_t, 6_d) \rightarrow 14_s.$$

Note that this mapping is not a homomorphism: $10_s + 6_s \neq 14_s$.

11. Divisible semi-groups. In the following definition, we make use of the fact that N acts on any semi-group.

DEFINITION 11A. *We say that the commutative semi-group $(G, +)$ is divisible if for each $x \in G$ and $n \in N$ there is a $y \in G$ such that $ny = x$. We say that $(G, +)$ is uniquely divisible if the above y is unique.*

Example 11B. The group N_m of integers mod m ($m > 1$) is not divisible; nor is N itself. The semi-groups Q^+ , R^+ , R are uniquely divisible. The group of dyadic rationals, containing all $k/2^n$ (k an integer, $n \in N$) is not divisible. The group of rationals mod 1 is divisible, but not uniquely; the same is true of the reals mod 1, or equivalently, of the multiplicative group of complex numbers of absolute value 1.

The direct sum of a finite set of uniquely divisible commutative semi-groups is uniquely divisible; in particular, any vector space over the reals is so.

DEFINITION 11C. *An element x of G is idempotent if $x + x = x$. We say G is idempotent if all its elements are idempotent.*

In a group, only the identity element is idempotent.

THEOREM 11D. *If G is idempotent, it is uniquely divisible.*

First, induction shows at once that $nx = x$ for all n and x . Now given x and n , set $y = x$; then $ny = x$. If also $ny' = x$, then $y' = ny' = x = ny = y$.

Note that, therefore, the operation of N on an idempotent semi-group is trivial.

Example 11E. Let U be a set, and let S be a set of subsets of U such that if A and B belong to S , so does their union $A \cup B$. Then (S, \cup) is an idempotent commutative semi-group.

REMARK 11F. It is a theorem that every idempotent commutative semi-group is of the form of Example 11E.

DEFINITION 11G. *Let us say that $(G, +)$ separates N if for any two distinct natural numbers m, n there is an $x \in G$ such that $mx \neq nx$.*

We cannot then reduce N to a smaller set of operators on G , as was done for instance in Section 7.

THEOREM 11H. *Any divisible commutative semi-group $(G, +)$ which is not idempotent separates N .*

Suppose not. Then for some m and n , $n = m + h$ and $mx = nx$ for all x . For some a and k , $m + a = kh$. Now take any $y \in G$. Choose x so that $kx = y$. Since $mx = (m + h)x$, adding ax gives $kx = (k + 1)hx$. Adding hx gives $(k + 1)hx = (k + 2)hx$. Continuing gives $kx = 2kx$, i.e. $y = 2y$. Thus G is idempotent, a contradiction.

12. Introduction of Q^+ . We assume here that $(G, +)$ is a uniquely divisible commutative semi-group which is not idempotent. Then, using Theorem 11H,

we have the properties

$$(12.1) \quad ax = ay \text{ implies } x = y \quad (a \in N; x, y \in G),$$

$$(12.2) \quad ax = bx \text{ for all } x \text{ implies } a = b \quad (a, b \in N; x \in G).$$

Given x and a , there is a unique y such that $ay=x$; let us denote this y by the expression $\bar{a}x$. Now, by definition,

$$(12.3) \quad a(\bar{a}x) = x \quad (a \in N; x \in G).$$

We may now form $a(\bar{b}x)$. We would like to write this in the form $(a\bar{b})x$. To this end, we must define new elements \bar{b} , and give the whole expression meaning.

First, we show that (writing \bar{b}' to denote $\overline{b'}$)

$$(12.4) \quad a(\bar{b}x) = a'(\bar{b}'x) \quad (\text{all } x \in G) \quad \text{iff} \quad ab' = a'b.$$

For, using (8.5) and (12.3) gives

$$\begin{aligned} (bb')(a(\bar{b}x)) &= (ab')(b(\bar{b}x)) = (ab')x, \\ (bb')(a'(\bar{b}'x)) &= (a'b)(b'(\bar{b}'x)) = (a'b)x. \end{aligned}$$

Now if the left hand side of (12.4) holds, then the above equations give $(ab')x = (a'b)x$ (all x), and $ab' = a'b$, by (12.2). The converse follows also, using (12.1).

Now consider the expressions $a\bar{b}$ as denoting new objects. Because of (12.4), we wish an equivalence relation between these objects:

$$(12.5) \quad a\bar{b} \sim a'\bar{b}' \quad \text{iff} \quad ab' = a'b.$$

(To prove that the relation is transitive, multiply $ab' = a'b$ by $a'b'' = a''b'$, giving $ab'a'b'' = a'ba''b'$, and apply the cancellation law in N , giving $ab'' = a''b$.) Denote the equivalence class of $a\bar{b}$ by a/b . Now let Q^+ be the set of equivalence classes thus obtained. (We could shorten $1\bar{b}$ to \bar{b} .) Note that $ac/bc = a/b$.

We now let Q^+ operate on G , by setting

$$(12.6) \quad \frac{a}{b}x = a(\bar{b}x) \quad (a, b \in N; x \in G).$$

Because of (12.4), the result is independent of the name a/b chosen for the given element of Q^+ .

Note that $\bar{1}x = x$; hence $(a/1)x = a(\bar{1}x) = ax$. We therefore *identify* the element $a/1$ of Q^+ with the element a of N , thus imbedding N in Q^+ , and the operation of N on G is preserved. (This is permissible; for (12.5) shows that $a/1 \neq b/1$ if $a \neq b$.)

If $\bar{b}x = \bar{b}y$, then $x = b(\bar{b}x) = b(\bar{b}y) = y$. Hence also, using (12.1),

$$(12.7) \quad rx = ry \text{ implies } x = y \quad (r \in Q^+; x, y \in G.)$$

The relation (12.4) gives:

$$(12.8) \quad rx = sx \text{ for all } x \in G \text{ implies } r = s \quad (r, s \in Q^+.)$$

13. Properties of Q^+ and G . First we note that for any $r=a/b \in Q^+$, using (8.5), $b(rx) = b(a(\bar{b}x)) = a(b(\bar{b}x)) = ax$. Hence

$$b[r(x+y)] = a(x+y) = ax + ay = b(rx) + b(ry) = b(rx+ry),$$

and applying (12.1) gives

$$(13.1) \quad r(x+y) = rx + ry \quad (r \in Q^+; x, y \in G).$$

Next, since

$$\frac{a}{b}x + \frac{c}{d}x = a(\bar{b}x) + c(\bar{d}x) = (a+c)(\bar{b}x) = \frac{a+c}{b}x,$$

we have

$$(13.2) \quad \frac{a}{b}x + \frac{c}{d}x = \frac{ad}{bd}x + \frac{bc}{bd}x = \frac{ad+bc}{bd}x.$$

Hence it is natural to define

$$(13.3) \quad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

Taking $b=d=1$ shows that this extends the definition in N . Now (13.2) gives the distributive law

$$(13.4) \quad (r+s)x = rx + sx \quad (r, s \in Q^+; x \in G).$$

That the definition of $r+s$ is independent of the manner of writing r and s may be verified directly; it also follows from (13.2) and (12.8) (using some G).

Since addition and multiplication in N are commutative, (13.3) shows that addition in Q^+ is commutative. Similarly, addition is associative. (Using G , these properties follow easily, on applying (12.8).) Hence Q^+ is a commutative semi-group.

From the definition of addition in Q^+ we have

$$\left(\frac{a}{b} + \frac{a}{b}\right) + \frac{a}{b} = \frac{2a}{b} + \frac{a}{b} = \frac{3a}{b},$$

and similarly, in general,

$$(13.5) \quad n \frac{a}{b} = \frac{na}{b}.$$

We can solve $ns=r$ for s : If $r=a/b$, set $s=a'/nb$. Thus Q^+ is divisible. If $n(a/b) = n(a'/b')$, then (13.5) and (12.5) give $na/b = na'/b'$, $nab' = na'b$, $ab' = a'b$, $a/b = a'/b'$; thus division (by elements of N) is unique. Clearly Q^+ is not idempotent.

Now Q^+ operates on itself; we call this operation multiplication. This extends the operation of multiplication in N . Because of (13.5),

$$bd \left(\frac{a}{b} \left(\frac{c}{d} \right) \right) = ad \left(\frac{b}{b} \left(\frac{c}{d} \right) \right) = ad \left(\frac{c}{d} \right) = ac = bd \frac{ac}{bd};$$

hence (12.1) gives

$$(13.6) \quad \frac{a}{b} \frac{c}{d} = \frac{ac}{bd}.$$

We now prove the general associative law:

$$(13.7) \quad r(sx) = (rs)x \quad (r, s \in Q^+; x \in G).$$

Say $r = a/b$, $s = c/d$. Then, with the help of (12.6) and (12.3),

$$bd \left(\frac{a}{b} \left(\frac{c}{d} x \right) \right) = ad \left(\frac{c}{d} x \right) = acx = bd \left(\frac{ac}{bd} x \right);$$

this gives the result.

In particular, multiplication in Q^+ is associative; it is clearly commutative. The distributive laws follow from (13.1) and (13.4). The cancellation law for multiplication follows from (12.7). The existence and uniqueness of division is clear: The solution of $(a/b)u = c/d$ is

$$(13.8) \quad u = \frac{c}{d} \div \frac{a}{b} = \frac{cb}{da}.$$

In particular, $1/(a/b) = b/a$.

The subtraction property in Q^+ is:

$$(13.9) \quad \text{If } r \neq s, \text{ then either } r+t=s \text{ or } s+t=r, \text{ for some } t.$$

For we may write $r = a/c$, $s = b/c$; then $a \neq b$, hence $b = a+d$ or $a = b+d$, and we may use $t = d/c$.

The cancellation law for addition is also easy to prove: Suppose $r+s = r+t$. Write $r = a/d$, $s = b/d$, $t = c/d$. Then $a+b = a+c$, hence $b = c$, and $s = t$.

The relation $r+s = r$ in Q^+ is impossible. Hence we may define order in Q^+ as in N (section 9), and Q^+ is now simply ordered.

CHAPTER II. RAYS

A "semi-ray" has certain properties needed for the measurement of positive quantities: addition and subtraction, order, and the existence of arbitrarily small elements. If the semi-ray is Archimedean, it may be completed to form a ray. The operation of Q^+ on a ray is extended to the operation by R^+ ; we find the basic properties of R^+ here.

14. Semi-rays. We shall not need divisibility in the definition; it will appear as a consequence of completeness in the next section.

DEFINITION 14A. A semi-ray L is a commutative semi-group such that:

(R₁) For all x and y in L , $x+y \neq x$.

(R₂) For all x and y in L with $x \neq y$, we can find u and v in L such that $x+u+v=y$ or $y+u+v=x$.

DEFINITION 14B. $x < y$ means that $x+u=y$ for some u .

Since we now have properties (a) and (b) of section 9, we may deduce the properties of *order* given there. We also have the *cancellation law*:

$$(14.1) \quad \text{If } x+u=y+u \text{ then } x=y.$$

For if $x \neq y$, then $x < y$ or $y < x$, by trichotomy, and hence $x+u < y+u$ or $y+u < x+u$, contrary to $x+u=y+u$.

We now introduce *subtraction*. If $x < y$, then there is a unique element u such that $x+u=y$; we call this element $y-x$. Now

$$(14.2) \quad (y-x)+x=y \quad \text{if } x < y.$$

We give some simple properties:

$$(14.3) \quad (x-y)-z=x-(y+z) \quad \text{if } y+z < x,$$

$$(14.4) \quad (x+y)-z = \begin{cases} x+(y-z) & \text{if } z < y, \\ x-(z-y) & \text{if } y < z < x+y. \end{cases}$$

To prove each one, add a quantity to each side which will remove the minus signs. Thus, for the last equality, add z :

$$\begin{aligned} [x-(z-y)]+z &= [x-(z-y)]+[(z-y)+y] \\ &= [(x-(z-y))+(z-y)]+y = x+y = [(x+y)-z]+z; \end{aligned}$$

now apply (14.1).

We now show that, owing to the particular form of (R₂), L contains arbitrarily small elements:

THEOREM 14C. For each $x \in L$ and $n \in N$ there is some $y \in L$ with $ny < x$.

First, $x+x \neq x$; also $x+x+u+v \neq x$ for all u and v , by (R₁); hence, by (R₂), we may find u and v so that

$$x+u+v=x+x.$$

By (14.1), $u+v=x$. Either $u \leq v$ or $v \leq u$; say $u \leq v$. Then $u+u \leq u+v=x$, and $u < 2u \leq x$; the statement holds with $n=1$. For the general case, we use induction: Say $nz < x$. Find y such that $2y \leq z$. Then

$$(n+1)y = ny + y \leq n(y+y) \leq nz < x.$$

DEFINITION 14D. The semi-ray L is Archimedean if for each x and y in L there is an $n \in N$ such that $y < nx$.

Example 14E. Let L consist of all ordered pairs $(x, 0)$ with $x \in R^+$, and (x, y) with $x \in R$ and $y \in R^+$ (or $y \in N$); define addition component-wise. Then $(L, +)$ is clearly a commutative semi-group, and (R_1) holds. To prove (R_2) , take $(x, y) \neq (x', y')$. If $y = y'$, then say $x < x'$; now $(x, y) + (x' - x, 0) = (x', y')$. If $y \neq y'$, say $y < y'$; then $(x, y) + (x' - x, y' - y) = (x', y')$. Now take $u = v = \frac{1}{2}(x' - x, 0)$ or $\frac{1}{2}(x' - x, y' - y)$. Since $n(1, 0) = (n, 0) < (0, 1)$ for all n , $(L, +)$ is not Archimedean.

15. Rays. We introduce completeness through Dedekind cuts.

DEFINITION 15A. An upper set U in a semi-ray L is a set such that

(U_1) $U \neq \emptyset$ and $U \neq L$.

(U_2) If $x \in U$ and $x < y$ then $y \in U$.

We say the upper set U is strict if also

(U_3) If $x \in U$ then there is some $y \in U$ with $y < x$.

DEFINITION 15B. Given $S \subset L$, z is a lower bound for S if $z \leq x$ for all $x \in S$. Also, z is a greatest lower bound (g.l.b. for short) for S if z is a lower bound, but no $z' > z$ is.

THEOREM 15C. Any subset of L has at most one g.l.b.

This is trivial.

DEFINITION 15D. The semi-ray L is complete if any strict upper set has a g.l.b.

THEOREM 15E. If L is complete, then any upper set has a g.l.b.

Suppose the upper set U is not strict. Then for some $z \in U$, $x < z$ implies $x \notin U$. Clearly z is the g.l.b. for U .

DEFINITION 15F. A ray is a complete semi-ray.

THEOREM 15G. A ray is Archimedean.

For suppose not. Say $nx \leq y$ for all n . Set

$$U = \{u: nx \leq u \text{ for all } n \in N\}.$$

Since $x \notin U$ and $y \in U$, (U_1) holds. Clearly (U_2) holds also; hence U is an upper set. By Theorem 15E, U has a g.l.b., say z . We show now that $z < nx$ for some n . If $z \leq x$, then $z < 2x$. Otherwise, $x < z$, and we may write $x + y = z$; now $y < z$ also. Therefore $y \notin U$, and we have $y < mx$ for some m ; hence $z = x + y < (m+1)x = nx$. But nx is a lower bound for U , contradicting the definition of z .

THEOREM 15H. Any ray L is uniquely divisible.

Take any x and any $n > 1$; suppose that $ny \neq x$ for all y . Set

$$U = \{u \in L: x < nu\}.$$

By Theorem 14C, $U \neq L$. Since $x \in U$, (U_1) holds. Using (9.5) in L shows that (U_2) holds; thus U is an upper set, and has a g.l.b., say z .

Suppose $nz < x$; say $nz + y = x$. Find h by Theorem 14C so that $nh < y$; then $n(z+h) < nz + y = x$ and (9.5) shows that $z+h$ is a lower bound for U , a contradiction. Hence $nz > x$, and we may write $x + y = nz$. Choose h so that $nh < y$. Now $nh < nz$, hence $h < z$, and we may write $v + h = z$; now

$$x + y = nz = n(v + h) < nv + y,$$

hence $x < nv$, and $v \in U$, contradicting $v < z$. We have proved that L is divisible. Since $z' < z$ implies $nz' < nz$, division is unique.

Since $x < x + x$, L is not idempotent. Hence (section 12) Q^+ operates on any ray L .

We give two facts about inequalities. For $r, s \in Q^+$ and $x, y \in L$,

$$(15.1) \quad r < s \quad \text{iff} \quad rx < sx,$$

$$(15.2) \quad x < y \quad \text{iff} \quad rx < ry.$$

If $r < s$, say $r + t = s$; then $rx + tx = sx$, and $rx < sx$. Similarly, if $r \geq s$, then $rx \geq sx$; hence if $rx < sx$ then $r < s$. (15.2) follows similarly.

Finally, we prove a *density* theorem:

$$(15.3) \quad \text{If } x, y, z \in L, x < y, \text{ then for some } r \in Q^+, x < rz < y.$$

Say $x + u = y$. For some $n \in N$, $nu > z$. Set $h = (1/n)z$; then $h < u$. Also $h < y$, and $mh > y$ for some $m \in N$. Hence, for some $k \in N$, $kh < y \leq (k+1)h$. Set $r = k/n$; then $rz = kh < y$. Also $x + u = y \leq rz + h < rz + u$, and hence $x < rz$.

16. Completion of an Archimedean semi-ray. Such a completion is a ray; in particular, we use this later to construct R^+ from Q^+ .

LEMMA 16A. *If L is a semi-ray and $x \in L$, then $U(x) = \{u \in L: x < u\}$ is a strict upper set.*

For $x \notin U(x)$, $2x \in U(x)$, and (U_1) holds. (U_2) is clear. Say $y \in U(x)$; then $x < y$. By (R_2) , we may write $x + u + v = y$. Now $x + u \in U(x)$ and $x + u < y$, so that (U_3) holds.

LEMMA 16B. *If U is an upper set in the Archimedean semi-ray L and $h \in L$, we may find $z \in L$ such that $z \notin U$ and $z + h \in U$.*

Choose $x \notin U$, $y \in U$. Say $nh > y$; then $x + nh \in U$. Hence, for z , we may take one of the elements $x, x + h, x + 2h, \dots, x + (n-1)h$.

LEMMA 16C. *If U and V are strict upper sets in the semi-ray L , then*

$$(16.1) \quad U + V = \{u + v: u \in U, v \in V\}$$

is a strict upper set; we have:

$$(16.2) \quad \text{If } u \notin U, v \notin V, \quad \text{then } u + v \notin U + V.$$

First, given $u \in U$, $v \in V$, take any $z \in U + V$, and write $z = u' + v'$, with $u' \in U$, $v' \in V$; then $u < u'$, $v < v'$, hence $u + v < z$, and $u + v \in U + V$.

Because of this, $U + V \neq L$. Clearly $U + V \neq \emptyset$. Properties (U_2) and (U_3) for $U + V$ are clear.

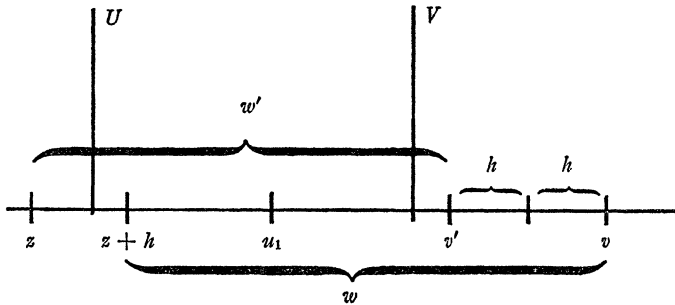
THEOREM 16D. *Let $(L, +)$ be an Archimedean semi-ray, let L^* be the set of all strict upper sets in L , and define addition in L^* by (16.1); then $(L^*, +)$ is a ray. The mapping $x \rightarrow U(x)$ of L into L^* (see Lemma 16A) is one-one and preserves addition; it thus imbeds L in L^* .*

With addition in L^* defined by (16.1), we clearly have a commutative semi-group. To prove (R_1) , take any strict upper sets U and V . Take $h \in V$, and find z for U by Lemma 16B. By (16.2), $z + h \in U + V$; since $z + h \in U$, $U \neq U + V$.

Next we introduce subtraction in L^* . Suppose $u_1 \in U$, $u_1 \in V$. Then set

$$W = \{w: \text{for some } w' < w, w' + u \in V \text{ for all } u \in U\}.$$

We have $V \subset W$; hence $W \neq \emptyset$. We may write $u_1 = u + h$, with $u \in U$; then $h \in W$, and $W \neq L$. Clearly (U_2) and (U_3) hold for W ; hence W is a strict upper set.



Clearly $U + W \subset V$. Conversely, take any $v \in V$. By (U_3) for V and Theorem 14C, we may write $v = v' + 2h$, with $v' \in V$. By Lemma 16B, find $z \in U$ such that $z + h \in U$. By (U_2) for U and for V , $z < u_1 < v'$; hence we may write $z + w' = v'$. Set $w = w' + h$. Now if $u \in U$, then $u > z$, and hence $w' + u > w' + z = v' \in V$; thus $w' + u \in V$. This shows that $w \in W$. Also $z + h \in U$, $w \in W$, and $z + h + w = v$; thus $v \in U + W$, proving that $U + W = V$.

Next, with W as above, choose x so that $2x \in W$. Since $2x \in U(x)$, we can find W' by the proof above so that $U(x) + W' = W$. Now we prove (R_2) . Given U and V with $U \neq V$, either $U \not\subset V$ or $V \not\subset U$, say the former. The proof above gives $U + U(x) + W' = V$, as required. Thus $(L^*, +)$ is a semi-ray.

To prove that L^* is complete, take any strict upper set $U^* \subset L^*$. The elements of U^* are strict upper sets in L ; let S be their union. Since $U^* \neq \emptyset$, $S \neq \emptyset$. Since $U^* \neq L^*$, there is a strict upper set $U \in U^*$. Take $x \in U$. For any $U' \in U^*$, $U < U'$ (in L^*), by (U_2) for U^* ; hence (Definition 14B) $U + W = U'$

for some W , and thus $x \notin U'$. This shows that $x \in S$, and $S \neq L$. Properties (U_2) and (U_3) for S are clear; thus S is a strict upper set in L , i.e. $S \in L^*$. Since $U \in U^*$ implies $U \subset S$ and hence $S \leq U$ in L^* , S is a lower bound for U^* in L^* . If $S < S'$ in L^* , then $S + W = S'$, and there is some $x \in S$ with $x \notin S'$. Now $x \in U$ for some $U \in U^*$, and hence $U < S'$ in L^* ; thus S' is not a lower bound for U^* . Therefore S is the g.l.b. of U^* in L^* , proving completeness.

Suppose $x < y$. Then say $x + u + v = y$; it follows that $x + u \in U(x)$, $x + u \in U(y)$, and $U(x) \neq U(y)$. Thus the mapping $x \rightarrow U(x)$ is one-one.

We must show still that $U(x + y) = U(x) + U(y)$. Suppose $z \in U(x + y)$; then $x + y < z$, and we may write $x + y + u + v = z$. Now $x + u \in U(x)$ and $y + v \in U(y)$, and thus $z \in U(x) + U(y)$. Thus $U(x + y) \subset U(x) + U(y)$. The converse is clear, and the proof is complete.

Suppose we form the completion L^* of the ray L . Each element U of L^* is a strict upper set in L ; it has a g.l.b., say x . The properties of strict upper sets show at once that $U = U(x)$ (compare Lemma 16A). Thus the mapping $x \rightarrow U(x)$ of L into L^* is onto, and we may identify L^* with L itself.

• **17. The isomorphism theorem.** We shall show that there is essentially only one kind of ray; any two rays are isomorphic. All homomorphisms are isomorphisms; we find them all. For further information, see Section 19.

DEFINITION 17A. Let $(G, +)$ and $(G', +)$ be commutative semi-groups. A mapping $\phi: G \rightarrow G'$ is a homomorphism if $\phi(x + y) = \phi(x) + \phi(y)$ for all x and y in G . It is an isomorphism if it is also one-one, and is onto G' if the image of G is all of G' .

THEOREM 17B. Let ϕ be a homomorphism of the ray L into the ray L' . Then

$$(17.1) \quad \text{if } x < y \text{ then } \phi(x) < \phi(y),$$

$$(17.2) \quad \text{if } r \in Q^+ \text{ then } \phi(rx) = r\phi(x).$$

If $x < y$, write $x + u = y$; then $\phi(x) + \phi(u) = \phi(y)$, and hence $\phi(x) < \phi(y)$. Next for any n , $\phi(x + \cdots + x) = \phi(x) + \cdots + \phi(x)$ (n terms in the sums); hence $\phi(nx) = n\phi(x)$. Also, if $y = (1/m)x$, i.e. $x = my$, then $\phi(x) = m\phi(y)$ and hence $\phi(y) = (1/m)\phi(x)$, from which (17.2) follows.

THEOREM 17C. Let L and L' be rays, and suppose $w \in L$, $w' \in L'$. Then there is a unique homomorphism ϕ of L into L' such that $\phi(w) = w'$. Any homomorphism of L into L' is an isomorphism onto L' .

For each $x \in L$, set

$$U'_x = \{y' \in L' : \text{for some } r \in Q^+, x < rw \text{ and } rw' < y'\}.$$

We see at once that U'_x is a strict upper set in L' ; hence (see the end of Section 16) for a certain $x' \in L'$, $U'_x = U'(x')$, with $U'(x')$ defined as in Lemma 16A. Set $\phi(x) = x'$.

We show that

$$(17.4) \quad U'_{x+y} = U'_x + U'_y.$$

If $z' \in U'_x + U'_y$, then $z' = x' + y'$, with $x' \in U'_x$, $y' \in U'_y$. For some $r, s \in Q^+$,

$$x < rw, \quad rw' < x'; \quad y < sw, \quad sw' < y'.$$

Now $x + y < (r + s)w$, $(r + s)w' < z'$, and hence $z' \in U'_{x+y}$. Conversely, given $z' \in U'_{x+y}$, we may find r, u, v so that $x + y + u + v = rw$, $rw' < z'$. By (15.3) we may find $s, t \in Q^+$ so that

$$x < sw < x + u, \quad y < tw < y + v.$$

Now $(s + t)w < rw$, and by (15.1), $s + t < r$, $(s + t)w' < rw' < z'$. We may write $(sw' + u') + (tw' + v') = z'$. Since $sw' + u' \in U'_x$ and $tw' + v' \in U'_y$, we have $z' \in U'_x + U'_y$, proving (17.4).

Recalling that $U'_z = U'(\phi(z))$ and $U'(x') + U'(y') = U'(x' + y')$ (see the end of Section 16), (17.4) gives

$$U'(\phi(x + y)) = U'_{x+y} = U'(\phi(x)) + U'(\phi(y)) = U'(\phi(x) + \phi(y));$$

hence $\phi(x + y) = \phi(x) + \phi(y)$, and ϕ is a homomorphism. Now we may apply (17.1), showing that ϕ is one-one.

Next we show that if ϕ' is any homomorphism of L' into L such that $\phi'(w') = w$, then $\phi(\phi'(x')) = x'$ for all $x' \in L'$. Set $x = \phi'(x')$, $y' = \phi(x)$. If $y' < x'$ we may find $r \in Q^+$ such that $y' < rw' < x'$. Then by Theorem 17B for ϕ' , $rw = \phi'(rw') < \phi'(x') = x$, and hence, applying ϕ , $rw < y'$, a contradiction. Similarly $x' < y'$ is false; hence $y' = x'$.

Because of this, ϕ maps L onto L' . Also, for another ϕ_1 like ϕ , we have $\phi_1(\phi'(x')) = x' = \phi(\phi'(x'))$, and since ϕ' is onto L , $\phi_1(x) = \phi(x)$ for all x ; thus ϕ is uniquely determined, and the proof is complete.

18. Introduction of R^+ . We saw in Section 13 that Q^+ is a completely divisible commutative semi-group; also, that $r + s = r$ is impossible in Q^+ , so that (R_1) holds. Given $r, s \in Q^+$, choose $t \in Q^+$ by (13.9); set $u = v = t/2$; then either $r + u + v = s$ or $s + u + v = r$, proving (R_2) . Therefore Q^+ is a semi-ray. Given $r, s \in Q^+$, we may write $r = a/c$, $s = b/c$; then since $ba \geq b$, (13.5) shows that $br \geq s$. Therefore Q^+ is Archimedean. We may now apply Theorem 16D, giving the completion R^+ of Q^+ ; R^+ is a ray, and we may consider Q^+ as imbedded in R^+ , by the definition $r \mapsto \{s \in Q^+ : r < s\}$.

19. The operation of R^+ on L . For each x in the ray L , let ϕ_x be the homomorphism of R^+ into L given by Theorem 17C, such that $\phi_x(1) = x$. By (17.2),

$$\phi_x(r) = \phi_x(r \cdot 1) = r\phi_x(1) = rx, \quad (r \in Q^+);$$

hence, if we set

$$(19.1) \quad ax = \phi_x(a) \quad (a \in R^+, x \in L),$$

this extends the operation of Q^+ to an operation of R^+ on L . Note that

$$(19.2) \quad 1x = x.$$

Since ϕ_x is a homomorphism, we have

$$(19.3) \quad (a + b)x = \phi_x(a + b) = \phi_x(a) + \phi_x(b) = ax + bx.$$

Given x and y in L , set $\psi(a) = \phi_x(a) + \phi_y(a)$ (all $a \in R^+$); then ψ is a homomorphism, and $\psi(1) = x + y$. By uniqueness in Theorem 17C, $\psi = \phi_{x+y}$. Hence

$$(19.4) \quad a(x + y) = \phi_{x+y}(a) = \psi(a) = \phi_x(a) + \phi_y(a) = ax + ay.$$

Since R^+ is a ray, R^+ operates on itself as above; let Φ_e be the corresponding functions. We call this operation *multiplication* in R^+ . Thus

$$(19.5) \quad ab = \Phi_b(a); \quad \Phi_b(1) = b.$$

Since the operation of R^+ on R^+ extends the operation of Q^+ on R^+ and hence of Q^+ on Q^+ , and the latter operation is multiplication in Q^+ , multiplication in R^+ extends that in Q^+ .

The above distributive laws give, in R^+ ,

$$(19.6) \quad (a + b)c = ac + bc, \quad a(b + c) = ab + ac.$$

Set $\Psi(a) = a$; Ψ is a homomorphism, and $\Psi(1) = 1$. By uniqueness, $\Psi = \Phi_1$. Hence $a1 = \Phi_1(a) = \Psi(a) = a$. Also, since $1 \in Q^+$ (or, by (19.5)), $1a = a$. Thus

$$(19.7) \quad 1a = a1 = a.$$

We can *divide* in R^+ : Given $a, b \in R^+$, since Φ_a is onto R^+ , we can find $c \in R^+$ such that $ca = \Phi_a(c) = b$.

Given $b \in R^+$ and $x \in L$, let ψ be the composite mapping $\phi_x \circ \Phi_b$; this is a homomorphism of R^+ into L . Since

$$\psi(1) = \phi_x(\Phi_b(1)) = \phi_x(b) = bx,$$

uniqueness shows that $\psi = \phi_{bx}$. Hence we find the general associative law:

$$(19.8) \quad a(bx) = \phi_{bx}(a) = \psi(a) = \phi_x(\Phi_b(a)) = \phi_x(ab) = (ab)x.$$

In particular, taking $L = R^+$,

$$(19.9) \quad a(bc) = (ab)c.$$

If we set $\Psi_b(a) = \Phi_a(b)$, then the second part of (19.6) shows that Ψ_b is a homomorphism. Since $\Psi_b(1) = b1 = b = \Phi_b(1)$, $\Psi_b = \Phi_b$ and

$$(19.10) \quad ba = \Phi_a(b) = \Psi_b(a) = \Phi_b(a) = ab.$$

THEOREM 19A. *The operation of R^+ on L extends the operation of Q^+ ; we have properties (19.1) through (19.10), and also:*

(V₁) *For each x and y in L there is an $a \in R^+$ such that $ax = y$.*

(V₂) *If $ax = ay$ then $x = y$.*

(V₃) *If $ax = bx$ then $a = b$.*

To prove (V₁), recall that ϕ_x is onto L ; hence for some a , $ax = \phi_x(a) = y$. To prove (V₂), set $c = 1/a$; then $ax = ay$ gives $c(ax) = c(ay)$, and by (19.8) and (19.2), $x = y$. Since ϕ_x is one-one, (V₃) holds.

We now extend (17.2):

THEOREM 19B. *If ϕ is a homomorphism of the ray L into the ray L' , then*

$$(19.11) \quad \phi(ax) = a\phi(x) \quad (a \in R^+, x \in L).$$

To show this, take any fixed $x \in L$, and set $\psi(a) = \phi(ax)$, $\theta(a) = a\phi(x)$. Using (19.3) in L and in L' shows that ψ and θ are homomorphisms of R^+ into L' . Since $\psi(1) = \theta(1)$, uniqueness in Theorem 17C shows that $\psi = \theta$, as required.

CHAPTER III. BIRAYS

A biray is constructed from a ray by adjoining a zero element and negative elements. The biray constructed from the ray R^+ is the group of real numbers; multiplication in R is defined through the operation of R on itself.

20. Definition of birays. Essentially, a biray B is a commutative group containing a ray B^+ , such that if $x \neq 0$ then x or $-x$ is in B^+ .

DEFINITION 20A. *A biray $(B, B^+, +)$ is a set B , a subset B^+ , and an operation of addition in B , such that:*

(B₁) $(B, +)$ is a commutative semi-group.

(B₂) $(B^+, +)$ is a ray.

(B₃) For each $x, y \in B$ there is a $z \in B$ such that $x + z = y$.

(B₄) If $x \neq y$, $x + z = y$, and $y + z' = x$, then $z \in B^+$ or $z' \in B^+$.

We prove first the *cancellation law*:

$$(20.1) \quad \text{If } x + z = x + z' \text{ then } z = z'.$$

For suppose $z \neq z'$. By (B₃), we may write $z + u = z'$, $z' + v = z$. By (B₄), one of u, v is in B^+ ; say $u \in B^+$. Now use (B₃) to write $x + z + w = u$. These relations give

$$u + u = x + z + w + u = x + z' + w = x + z + w = u.$$

But $u \in B^+$, contrary to (B₂) and (R₁) (see Section 14).

We now find the *zero* element of B . Choose $x_0 \in B^+$. Choose $0 \in B$ so that $x_0 + 0 = x_0$, by (B₃). We show that for all $x \in B$,

$$(20.2) \quad x + 0 = x.$$

By (B₃), we may write $x = x_0 + y$. Now $x + 0 = y + x_0 + 0 = y + x_0 = x$. Moreover,

$$(20.3) \quad \text{if } x + u = x \text{ then } u = 0,$$

by the cancellation law; hence the zero element 0 is uniquely defined by (20.2).

We now know that $(B, +)$ is a commutative group.

DEFINITION 20B. *For $x \in B$, $-x$ is the element such that*

$$(20.4) \quad x + (-x) = 0;$$

set also

$$(20.5) \quad x - y = x + (-y).$$

We have

$$(20.6) \quad (x - y) + y = x + [(-y) + y] = x + 0 = x.$$

Using this gives

$$(20.7) \quad y = z - x \quad \text{iff} \quad x + y = z$$

(add x in the first equation; add $-x$ in the second). Also

$$(20.8) \quad -0 = 0, \quad -(-x) = x;$$

for $0+0=0$; also $(-x)+x=0$, $(-x)+(-(-x))=0$, and the cancellation law gives the second part of (20.8). Some further elementary properties are:

$$(20.9) \quad -(x + y) = (-x) + (-y) = -x - y,$$

$$(20.10) \quad -(x - y) = y - x,$$

$$(20.11) \quad (x - z) + (y - w) = (x + y) - (z + w).$$

We may prove each directly from (20.4) and (20.5). Or, we may add something to both sides and apply (20.1). For instance,

$$-(x - y) + x = -(x - y) + [(x - y) + y] = 0 + y = y,$$

and also $(y - x) + x = y$; hence (20.10) follows.

21. Order in birays. We first consider negative elements.

DEFINITION 21A. B^- is the set of all x such that $-x \in B^+$. The elements of B^+ are positive; those of B^- are negative.

We prove trichotomy: Each element of B is zero, positive or negative, and is only one of these.

Given x , at least one holds; for if $x \neq 0$, then since $x + (-x) = 0$ and $0 + x = x$, applying (B_4) shows that x is in B^+ or in B^- . At most one is true. For, since $(B^+, +)$ satisfies (R_1) (Section 14), and $0 + 0 = 0$, $0 \notin B^+$. Also, if $x \in B^- \cap B^+$, then $-x$ and x are in B^+ , and by (B_2) , $0 = x + (-x) \in B^+$, a contradiction.

DEFINITION 21B. $x < y$ means that $y - x \in B^+$.

Because of (B_2) , order is transitive. The trichotomy above is equivalent to trichotomy in terms of order (Section 9). Thus B is simply ordered. The properties (9.2), (9.3) and (9.4) clearly hold. Note that

$$(21.1) \quad -x < -y \quad \text{iff} \quad y < x.$$

One may introduce *absolute values* in terms of order. Note that if $x \leq y$ and $y \leq x$, then $x = y$. Hence the following definition makes sense:

$$(21.2) \quad \text{If } x \leq y, \text{ then set } \inf\{x, y\} = x, \quad \sup\{x, y\} = y;$$

set

$$(21.3) \quad |x| = \sup \{x, -x\}.$$

The derivation of properties of absolute value is standard; we need not go into it here. The definitions of $\inf \{x_1, \dots, x_n\}$, $\sup \{x_1, \dots, x_n\}$ are clear.

22. The isomorphism theorem. We prove:

THEOREM 22A. *Let $(B, B^+, +)$ and $(B', B'^+, +)$ be birays. Take any $w \neq 0$ in B and any w' in B' . Then there is a unique homomorphism ϕ of $(B, +)$ into $(B', +)$ such that $\phi(w) = w'$. If $w' \neq 0$, then ϕ is an isomorphism onto B' . If w and w' are both positive or both negative, then ϕ carries B^+ onto B'^+ and B^- onto B'^- , and*

$$(22.1) \quad \phi(x) < \phi(y) \quad \text{iff} \quad x < y;$$

if one of w, w' is positive and the other negative, ϕ has the opposite effect.

See also Theorem 24A.

Suppose first that w and w' are positive. By Theorem 17C, there is a unique homomorphism ϕ^+ of B^+ into B'^+ such that $\phi^+(w) = w'$; hence the restriction $\phi|_{B^+}$ of ϕ to B^+ must be ϕ^+ . We must have $\phi(x) = \phi(x+0) = \phi(x) + \phi(0)$, and hence $\phi(0) = 0$ (letting 0 denote the zero element in both birays). Also we must have $\phi(x) + \phi(-x) = \phi(x+(-x)) = \phi(0) = 0$, and hence $\phi(-x) = -\phi(x)$. Thus ϕ , if it exists, is unique. We may use these equations to define ϕ outside B^+ .

To prove that ϕ is a homomorphism, we examine $\phi(\alpha + \beta)$ for any $\alpha, \beta \in B$. The case that $\alpha = 0$ or $\beta = 0$ is clear, and the case $\alpha, \beta \in B^+$ is known. The remaining cases are as follows: For $x, y \in B^+$,

$$\begin{aligned} \phi(-x + (-y)) &= \phi(-(x + y)) = -\phi(x + y) = -[\phi(x) + \phi(y)] \\ &= -\phi(x) - \phi(y) = \phi(-x) + \phi(-y); \end{aligned}$$

$$\phi(x + (-x)) = \phi(0) = 0 = \phi(x) - \phi(x) = \phi(x) + \phi(-x);$$

$$\text{if } y < x \text{ then } \phi(x + (-y)) = \phi(x - y) = \phi(x) - \phi(y) = \phi(x) + \phi(-y);$$

$$\begin{aligned} \text{if } x < y \text{ then } \phi(x + (-y)) &= \phi(-(y - x)) = -\phi(y - x) \\ &= -[\phi(y) - \phi(x)] = \phi(x) + \phi(-y); \end{aligned}$$

also $\phi((-x) + y) = \phi(y + (-x))$, and the above applies.

Since ϕ^+ is an isomorphism onto B'^+ , ϕ carries B^+ onto B'^+ and B^- onto B'^- . Thus ϕ is onto B' . That ϕ is also one-one and is therefore an isomorphism is clear. If $x < y$, say $x + z = y$, $z \in B^+$. Expanding $\phi(x + z)$ shows that (22.1) holds.

Next suppose that w is positive and w' is negative. Set $w'' = -w'$, and let ϕ' be the homomorphism of B into B' with $\phi'(w) = w''$; then we see easily that $\phi(x) = -\phi'(x)$ is the unique required isomorphism.

If w is positive and $w' = 0$, we show that $\phi(w) = 0$ for all $x \in B$. If $x \in B^+$ and $\phi(x) \in B^+$, then by Theorem 17C, $\phi(y) \in B^+$ for all $y \in B^+$, contrary to $\phi(w) = 0$. If $x \in B^+$ and $\phi(x) \in B^-$, then $\phi' = -\phi$ is a homomorphism with $\phi'(w) = 0$ and

$\phi'(x) \in B^+$, and we again have a contradiction. Thus $\phi(x) = 0$, all $x \in B^+$. The case that $x \in B^-$ is similar.

Finally, if w is negative, using the pair $(-w, -w')$ in place of (w, w') gives again the required properties.

23. Construction of a biray from a ray. Given a ray $(B^+, +)$, we construct a corresponding biray $(B, B^+, +)$. The elements of B are those of B^+ , called positive, a new element 0, and for each element x of B^+ , a new element x^* ; the latter form B^- , the negative elements of B . We shall use x, y, \dots to denote elements of B^+ , and α, β, \dots to denote elements of B (including B^+). We define addition in B as follows:

$$(23.1) \quad \alpha + 0 = \alpha,$$

$$(23.2) \quad x + x^* = 0,$$

$$(23.3) \quad x + y^* = \begin{cases} x - y & \text{if } y < x, \\ (y - x)^* & \text{if } x < y, \end{cases}$$

$$(23.4) \quad x^* + y^* = (x + y)^*;$$

requiring addition to be commutative gives the remaining cases.

To prove properties of addition it is convenient to introduce new names for elements of B (corresponding to the "ordered pair" definition of negative numbers): For any $x, y \in B^+$, set

$$(23.5) \quad [x, y] = x + y^*.$$

As a consequence, for $x, y, z \in B^+$,

$$(23.6) \quad [y + z, y] = z, \quad [x, x] = 0, \quad [x, x + z] = z^*;$$

this shows that each element of B has new names.

We show now that

$$(23.7) \quad [x, y] = [x, y'] \quad \text{iff} \quad y = y'.$$

Suppose $[x, y] = [x, y']$. From (23.5) and trichotomy we see first that whichever case in (23.2), (23.3) applies to x and y also must apply to x and y' ; now if $y < x$, $x - y = x - y'$ implies $y = y'$, and the situation is similar in the other cases.

Applying (20.11) to (23.2) or (23.3) shows that

$$(23.8) \quad [x + z, y + z] = [x, y].$$

Next, since $[x, y] = [x + u, y + u]$ and $[u, v] = [x + u, x + v]$, (23.7) gives

$$(23.9) \quad [x, y] = [u, v] \quad \text{iff} \quad x + v = y + u.$$

We are now ready to prove that

$$(23.10) \quad [x, y] + [u, v] = [x + u, y + v].$$

For $[x, y]$ there are three cases to consider: $[y + z, y]$, $[x, x]$ and $[x, x + z]$; similarly for $[u, v]$. Some of these cases are:

$$\begin{aligned}
[x, x] + [u, v] &= 0 + [u, v] = [u, v] = [x + u, x + v], \\
[y + z, y] + [v + w, v] &= z + w = [y + z + v + w, y + v], \\
[x, x + z] + [u, u + w] &= z^* + w^* = (z + w)^* = [x + u, x + z + u + w],
\end{aligned}$$

if $w < z$ then

$$\begin{aligned}
[y + z, y] + [u, u + w] &= z + w^* = z - w = [y + (z - w), y] \\
&= [y + z + u, y + u + w],
\end{aligned}$$

if $w < z$ then

$$\begin{aligned}
[y + z, y] + [u, u + w] &= (w - z)^* = [y, y + (w - z)] \\
&= [y + z + u, y + u + w]; \\
[y + z, y] + [u, u + z] &= z + z^* = 0 = [y + z + u, y + u + z];
\end{aligned}$$

the remaining cases are taken care of by commutativity.

Applying (23.10) twice gives

$$([x, y] + [u, v]) + [p, q] = [(x + u) + p, (y + v) + q];$$

now associativity in B^+ gives associativity in B . We now have (B_1) and (B_2) for $(B, B^+, +)$.

To prove (B_3) , we need merely note that

$$(23.11) \quad [x, y] + [u + y, v + x] = [x + u + y, y + v + x] = [u, v].$$

Next we prove the cancellation law, through the implications

$$\begin{aligned}
[x, y] + [u, v] &= [x, y] + [p, q] \Rightarrow [x + u, y + v] = [x + p, y + q] \\
&\Rightarrow x + u + y + q = y + v + x + p \Rightarrow u + q = v + p \Rightarrow [u, v] = [p, q].
\end{aligned}$$

To prove that (B_4) holds, suppose that

$$[x, y] \neq [u, v], \quad [x, y] + \alpha = [u, v], \quad [u, v] + \beta = [x, y].$$

By (23.11) and the cancellation law, we then have

$$\alpha = [u + y, v + x], \quad \beta = [x + v, y + u].$$

Now by (23.9), $x + v \neq y + u$; hence either $x + v < y + u$ and $\alpha \in B^+$ or $x + v > y + u$ and $\beta \in B^+$.

We have now proved that $(B, B^+, +)$ is a biray.

REMARK: An alternative proof may be given as follows: First show that

$$\text{if } z + \alpha = z + \beta, \quad z > |\alpha| + |\beta|, \quad \text{then } \alpha = \beta;$$

this is easy. Next, considering several cases, we find:

$$\text{If } z > |\alpha| + |\beta| \quad \text{then} \quad z + (\alpha + \beta) = (z + \alpha) + \beta.$$

Using these gives at once:

If $z > |\alpha| + |\beta| + |\gamma|$ then $z + [(\alpha + \beta) + \gamma] = z + [\alpha + (\beta + \gamma)]$;

this gives the associative law. To prove (B_3) , use $x + (x^* + \beta) = \beta$, $x^* + (x + \beta) = \beta$. Finally, (B_4) is proved by considering several cases.

There is essentially only one biray containing a given ray:

THEOREM 23A. *If $(B, B^+, +)$ and $(B', B^+, +)$ are birays, then there is an isomorphism ϕ of the first onto the second, defined by $\phi(0) = 0'$, and $\phi(x) = x$, $\phi(-x) = -x$, ($x \in B^+$).*

Here, the first " $-x$ " is interpreted in B , and the second, in B' ; see Definition 20B. The theorem follows at once from Theorem 22A, choosing some $w = w'$ in B^+ .

24. The operation of R on a biray. First, define $(R, R^+, +)$ to be the biray constructed from $(R^+, +)$ as in Section 23; by Theorem 23A, this is (up to isomorphisms) the only biray containing $(R^+, +)$ as its positive part.

We now define the operation of R on any biray $(B, B^+, +)$. For each $\alpha \in B$, let ϕ_α be the homomorphism of $(R, R^+, +)$ into $(B, B^+, +)$ given by Theorem 22A, with $\phi_\alpha(1) = \alpha$; set

$$(24.1) \quad a\alpha = \phi_\alpha(a) \quad (a \in R, \alpha \in B).$$

This extends the operation of R^+ on the part B^+ of B . Exactly as in Section 19, we find (always using $a, b, \dots \in R$; $\alpha, \beta, \dots \in B$)

$$(24.2) \quad (a + b)\alpha = a\alpha + b\alpha, \quad a(\alpha + \beta) = a\alpha + a\beta.$$

Since $(R, R^+, +)$ is a biray, it operates on itself; this operation we call *multiplication* in R . Now

$$(24.3) \quad (a + b)c = ac + bc, \quad a(b + c) = ab + ac.$$

There is no harm in identifying the zero element in different birays; in particular, in R and in B . Since ϕ_α is a homomorphism, $\phi_\alpha(0) = 0$. We now have (compare Section 19)

$$(24.4) \quad 1\alpha = \alpha, \quad 0\alpha = \alpha 0 = 0,$$

$$(24.5) \quad 1a = a1 = a, \quad 0a = a0 = 0.$$

Again as in Section 19,

$$(24.6) \quad a(b\alpha) = (ab)\alpha, \quad a(bc) = (ab)c, \quad ab = ba.$$

We now show that we can *divide* in R . Given $a, b \in R$ with $a \neq 0$, the homomorphism Φ_a of R into itself with $\Phi_a(1) = a$ is an isomorphism onto R (Theorem 22A); hence for some $c \in R$, $ca = \Phi_a(c) = b$.

THEOREM 24A. *If ϕ is a homomorphism of the biray B into the biray B' , then*

$$(24.7) \quad \phi(a\alpha) = a\phi(\alpha) \quad (a \in R, \alpha \in B).$$

The proof of Theorem 19B applies, using Theorem 22A (which holds also for the case that $\phi(\alpha) \equiv 0$).

THEOREM 24B. *$(R, +, \times)$ is a complete ordered field, unique up to isomorphisms.*

We have proved that we have a field; it is ordered. Since R^+ is complete, the proof that R is complete (using Dedekind cuts) is simple. Suppose $(R', +, \times)$ is another such field. Then $1 \in R^+$, $1' \in R'^+$. Let ϕ be the corresponding homomorphism of birays, with $\phi(1) = 1'$; this is an isomorphism onto R' , by Theorem 22A.

Now take any fixed b , and set

$$\psi(a) = \phi(ab), \quad \theta(a) = \phi(a)\phi(b).$$

Then

$$\begin{aligned} \psi(a_1 + a_2) &= \phi((a_1 + a_2)b) = \phi(a_1b + a_2b) \\ &= \phi(a_1b) + \phi(a_2b) = \psi(a_1) + \psi(a_2), \end{aligned}$$

and similarly θ is a homomorphism. Since $\theta(1) = 1'\phi(b) = \phi(b) = \psi(1)$, Theorem 22A shows that $\psi = \theta$; that is, ϕ is a multiplicative homomorphism. Since $\phi(1) = 1'$, ϕ maps R^+ onto R'^+ . Thus ϕ is an isomorphism of ordered fields.

Note that ϕ is the only such isomorphism; for we must have $\phi(1) = \phi(1 \cdot 1) = \phi(1)\phi(1)$, and since $\phi(1) \neq 0'$, $\phi(1) = 1'$.

THEOREM 24C. *For any biray B , we have:*

(V_1^*) *For each α and β in B , $\alpha \neq 0$, there is an $a \in R$ such that $a\alpha = \beta$.*

(V_2^*) *If $a\alpha = a\beta$, $a \neq 0$, then $\alpha = \beta$.*

(V_3^*) *If $a\alpha = b\alpha$, $\alpha \neq 0$, then $a = b$.*

The proofs are like the corresponding proofs in Theorem 19A; we use the fact that if $\alpha \neq 0$ then ϕ_α is onto B and is one-one.

THEOREM 24D. *Any biray $(B, B^+, +)$, with the operation of R , is an oriented one-dimensional vector space over R , and conversely.*

By (24.2), (24.6) and (24.4), the biray is a vector space; it is oriented by the choice of B^+ as "positive" part. Because of (V_1^*) , it is one-dimensional. Conversely, any such space satisfies the postulates for a biray, and the operation by R is the same as that defined here.