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ORIGINAL ARTICLE

Reform of the construction of the number system with reference to Gottlob Frege

Heinz Griesel

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Abstract Due to missing ontological commitments Frege rejected Hilbert's Fundamentals of Geometry as well as the construction of the system of real numbers by Dedekind and Cantor. Almost all of school mathematics is ontologically committed. Therefore, H.-G. Steiner considered Frege's viewpoint of mathematics fundamentals, refined by Tarski's semantics, as suitable for math education. Frege committed numbers ontologically by using measurement to define numbers. He invented the concept of quantitative domain (Größengebiet), which – it is now known by reconstruction of that concept by the New-Fregean Movement – agrees with the concept of quantity domain (Größenbereich) as established in the German reform of the application-oriented construction of the system of real numbers. Concepts of quantity (ratio-scale) and interval-scale in comparative measurement theory – going beyond Frege – show the way how the negative numbers can be ontologically committed and the operations of addition and multiplication can be included. In this work it is shown how Frege's viewpoint of mathematics fundamentals, as propagated by H.-G. Steiner, can be better implemented in the current construction of the system of real numbers in school.

H. Griesel (✉)
Fachbereich Mathematik/Informatik,
Universität Kassel, Postfach 10 13 80,
34109 Kassel, Germany
e-mail: hgriesel@web.de

1 Appropiate foundational viewpoint for math education: objective of this work

In 1964, Steiner (1964, 1965a) pointed out a correspondence between Frege and Hilbert, commenting on its essence in detail. Frege criticizes Hilbert's foundations of geometry. Using a contemporary philosophical term, Frege's critique can be summarized as follows: There is a lack of *ontological commitments*, meaning there is no connection between geometry and reality. Frege even claimed that ontological commitments are a substantial component of all mathematics. In school mathematics ontological commitments may be absent in only a few places. They are anyhow essential for application-oriented mathematics instruction. In this context Steiner (1965a, p. 45) states: ''Frege's viewpoint, refined by modern semantics, can serve excellently as the scientific theoretical background''. (''Als wissenschaftstheoretischer Hintergrund hierzu kann in ausgezeichneter Weise der durch die moderne Semantik verfeinerte Standpunkt Freges dienen''). Steiner considered Tarsky's semantics as refinement. Also, in the disagreement with his PhD mentor, Laugwitz, he expressed the same opinion (Steiner, 1965b). He observed this standpoint implemented in the modern theory of equations, to which he had provided important aspects (Steiner, 1961).

One of the central topics of school mathematics is the construction of the number system. It is, like most of the other parts of school mathematics, ontologically committed. Therefore, it is the main objective of this work to bring together Frege's consideration with the German reforms of the construction of the number system which were established in the 1960s and 1970s without reference to Frege.

In this context we have to question if all elements of Frege's construction of the number system are really ontologically committed and if they are conceptually uniform. We have also to explore the weaknesses of the aforementioned German projects and to find homogeneous aspects of the construction of the number system in school.

2 Frege's application-oriented construction of the real number system

2.1 Frege's critique of Dedekind's and Cantor's construction of the real number system

It is too little known that Frege with similar arguments as against Hilbert refuted Dedekind's and Cantor's construction of the number system. He raised his main criticism demanding that real numbers used for measuring should be involved in their definition (Kutschera, 1989, p. 120). Frege writes literally for his own intentions (Frege, 1903, p. 157): "We simultaneously avoid the deficiency that the measuring either does not occur at all or that measuring and numbers are patched together without internal consistency reasonable in the essence of the number itself." ("Wir vermeiden zugleich den bei diesen hervortretenden Mangel, dass das Messen entweder gar nicht vorkommt oder ohne inneren im Wesen der Zahl selbst begründeten Zusammenhang rein äußerlich angeflickt wird.") Obviously Frege regarded measuring as the means with which the ontological commitments of the numbers are produced. In fact, Dedekind and Cantor did not reference measuring at all. The construction of the number systems by both, and later by Landau (1930) as well as in the excellent book from Oberschelp (1968) all took place without ontological commitments.

2.2 Frege's own theory of the construction of the real number system

Unlike geometry, Frege attempted to fulfill the demand for ontological commitments during his construction of the real number system. The theory of the natural numbers is represented in the book *The Foundations of Arithmetic* (Frege, 1884), with further construction of the number system in the two-volume work, *The Basic Laws of Arithmetic* (Frege, 1893, 1903). The natural numbers are introduced as finite cardinals while the positive rational and real numbers are coefficients of measure. In this way a clear conceptual difference between both kinds of numbers appears, which we will be preoccupied with later.

In order to include the measuring according to his own demand of ontological commitments, Frege was the first to have the idea to introduce the concept of *quantitative domain*. He writes (Frege, 1903, p. 159): "Something is a quantity not alone for itself but only if it is an element with other objects of a class that is a *quantitative domain*." ("Etwas ist eine Größe nicht für sich allein, sondern nur, sofern es mit anderen Gegenständen einer Klasse angehört, die ein *Größengebiet* ist.") Lengths, areas, volumes, durations, masses and angles form quantitative domains.

Frege's considerations about the definition of numbers are difficult to comprehend because Frege uses the two-dimensional *Begriffsschrift* (concept-script) for the formalization of the mathematical facts introduced by himself, which is not familiar to contemporary mathematicians and because Frege's position is concept-theoretical rather than set-theoretical. Furthermore, his considerations are incomplete. Originally he had planned another third volume that was not published after Russell discovered a contradiction in Frege's logical system now known as Russell's paradox (Russellsche Antinomie).

2.3 The reconstruction of Frege's considerations in the new-Fregean movement

Since some years there has been an attempt to revive Frege's considerations (Demopolous, 1995; Hale and Wright, 2001;). Main representatives of this Neo-Fregeanism are the philosophers Crispin Wright (St. Andrews) and Bob Hale (formerly Glasgow, now Sheffield). Thereby, Frege's construction of the number system was reconstructed with the usual means of structural mathematics. The concept of quantitative domain introduced by Frege is defined as a linear-ordered, Abelian semigroup, which suffices the further condition $a < b \Leftrightarrow \exists c \; a + c = b$ (Hale, 2001, p. 405; Stein, 1995, p. 339).

2.4 Deficits in Frege's construction of the number system

As previously mentioned, Frege did not have any uniform concept of number. That is beyond doubt a first deficit. Furthermore, Frege had insufficiently fulfilled his own demand to involve measuring in the definition of numbers. One can consider this demand fulfilled for the natural and the positive rational numbers. The negative numbers on the other hand are introduced without involving measuring and, therefore, without ontological commitments. Additionally, Frege neglected to consequently meet the stipulations of including measuring in the definitions of the introduced operations of addition and multiplication. Both definitions are defined free from measuring and therefore are cut off from ontological commitments.

Hence, three deficits appear in the application-oriented construction of the number system according to Frege:

– There is no uniform number concept.
– The negative numbers are not defined by involving measuring.
– The operations addition and multiplication are not defined by involving measuring.

2.5 Rehabilitation of the principles of abstraction by Neo-Fregeanism

The Neo-Fregeanism has also brought a rehabilitation of the *abstraction principles* (also called *definitions by abstraction*) and that they can be articulated as legitimate foundations for mathematical theories. That means: If ~ is an equivalence relation, it is possible to proceed mentally to that abstraction which all elements of an equivalence class have in common and mentally act with this common property $E(x)$ of the element x as a mathematical object (cf. the full discussion in Fine, 1998; about 100 pages). It is then valid: $E(x) = E(y) \Leftrightarrow x{\sim}y$

These properties as abstractions must be treated as objects in their own right. In modern mathematics it has been customary to identify these abstractions with the equivalence classes. That is in general not practical in math education.

In school mathematics, these principles of abstraction are unconsciously used in the quantities *length*, *area*, *volume*, *duration*, *mass* and the *finite cardinals*. Each of these quantities has a set of objects, in which an equivalence relation is defined.

To the quantity length appertain as objects the *line segments*. The equivalence relation is *congruent*.

The quantity *area* has surfaces as objects. The equivalence relation is *equidecomposable* (decomposed into parts so that the parts are two-by-two congruent, in German: *zerlegungsgleich*).

The quantity *duration* has *events* as objects. The equivalence relation is *takes as long as.*

The quantities volume and mass have the same objects, namely *bodies*. But, the equivalent relations are different. In the case of volume, there is the equivalence relation *equivoluminous*. A equivoluminous B means that when fluid completely fills A or B it is possible to pour fluid from A to B and from B to A without any spillage or when A and B are immersed into a body of fluid, the fluid level is equally raised each time.

The quantity mass has the equivalence relation *equimassive*. A equimassive B means that there is balance if the bodies A and B are on the balance scales.

The quantity *finite cardinal* has *finite sets* as objects. The equivalence relation is *equinumerous* (there is a one-to-one correspondence). This abstraction principle is also called *Hume's Principle*. It was already used by Frege.

In the next step, one goes in school mathematics mentally from these equivalence relations to the abstractions, which have in common the equivalent objects in each case. These abstractions are called the *quantity values*, respectively, in the special cases *length-values, area-values, volume-values, duration-values, mass-values, finite cardinal-values*.

The quantities, therefore, are interpreted as special mappings with a set of objects as *domain*. The values of the quantity form the *range* of that mapping. The mapping is also called a *scale*. It is valid for instance for the quantity *length* L: $L(s) = L(s') \Leftrightarrow s$ congruent s' and for the quantity *finite cardinal* N: $N(A) = N(B) \Leftrightarrow A$ equinumerous B

Frege himself has used as an example the *lines* with the equivalence relation *parallel*. The common abstraction of parallel lines is the *direction of the lines*. The direction of lines is not a quantity. It is a *nominal-scale*.

3 Reforms of the construction of the number system in German school mathematics

3.1 Scientific orientation applied to the construction of the number system

In 1960–1970s, Germany took part in the worldwide curriculum reform of mathematical instruction. An important feature of the reform was the scientific orientation of the curricula, that is how contemporary mathematics should be presented. Under this point of view, there was also a question of the construction of the number system, in particular the calculation with fractions, that is, the introduction of the positive rational numbers and their operations addition and multiplication. Nevertheless, the mathematics didacticians of the Federal Republic of Germany agreed that a very strict scientific orientation as dictated in the GDR was educationally not reasonable. The construction of the number system must indeed be mathematically well-founded, but furthermore genetic, oriented at the developmental and thinking level of the students, and application-oriented from the beginning (Griesel, 2001).

3.2 Positive rational numbers as operators on quantity domains

The mathematical foundations of the reform of the construction of the number system were researched by Lugowski (1962), Kirsch (1970), Pickert (1968) and Griesel (1968, 1971, 1973a, b, 1974). In order to achieve application-orientation, Lugowski (1962) introduced, without knowledge of Frege's considerations, the concept of *quantity domain*. The definition, going back to Kirsch, of a quantity domain as a linear-ordered, Abelian semi-group, with the additional condition $a < b \Leftrightarrow \exists c\ a + c = b$ is now used (Kirsch, 1970, p. 43). This was developed independent of Frege's deliberations 30 years before the reconstruction of Frege's considerations by the Neo-Fregeanism. It is remarkable that the concept of *quantity domain* according to Kirsch agrees with the concept of *quantitative domain* according to Frege.

Also Hans Georg Steiner pursued these considerations with interest although he himself has not intervened actively. After all, a report about this German development in *Educational Studies* (Steiner, 1969) was written by him.

The positive rational numbers were introduced as operators on divisible quantity domains (Griesel, 1973a). For instance, the number ¾ became a chain of the stretcher ·3 with the shrinker :4. That had the advantage of machanical concretization. The forming of fraction-parts could be regarded as applying an operator to a value of a quantity and multiplication as connecting fraction operators in series dynamically.

3.3 Problems with the implementation of the curricula

During the implementation of the curricula there were difficulties in understanding in the majority of the classes. The embedding of the natural numbers into the new rational numbers was not understood by many students. That was not surprising since the students were acquainted with the natural numbers for 5 years as finite cardinals. Now these numbers should abruptly be operators. Also, the proposal of Kirsch to interpret the natural numbers in a specific curriculum as operators before beginning with the introduction of the rational numbers was not successful (Griesel, 1973a). Therefore, the introduction of the natural numbers as operators was stopped. The authors of the textbooks returned to a more conventional introduction. But, they were at least able to improve their curricula by some results of didactical research like the *quasi-cardinal aspect* of the positive rational numbers (see Sect. 3.4).

3.4 Basic structure of the current curricula for the construction of the number system

The current curricula for the construction of the number system in Germany have the following basic structure: From the first to fifth school years, the natural numbers are regarded as finite cardinals, therefore, as values of a quantity. Addition and multiplication are also introduced in connection with this quantity. A detachment from the idea of finite cardinals, and therefore, a treatment of natural numbers as objects in their own right takes place in the theory of divisibility in the fifth school year with the concepts *divisor, multiple, greatest common divisor, least common multiple*, and *prime number*.

In the theory of fractions in the sixth school year the positive rational numbers are understood as *ratios of a part to a whole* (in German: *Anteile*) and therefore also as values of a quantity. Besides the difficulties, given by the concepts of natural numbers as finite cardinals and the positive rational numbers as ratios of a part to a whole are reduced by quasi-cardinal considerations (Griesel, 1981). For instance, a fraction like ¾ is in this case understood as 3 fourths, therefore, as a cardinal of fourths. But it is not possible to give reason for the rule of reducing fractions to lower respective higher terms by the quasi-cardinal aspect except by the intuitive concept of ratio of a part to a whole.

The picture given by the negative numbers is not so homogenous. There are several manners and forms of the negative numbers in the curricula. References are found in Postel (2005) and Griesel (2003).

There are also no homogenous characteristics of the operations addition and multiplication that can be found in all kinds of numbers.

The multiplication of natural numbers is introduced in close relationship to the interpretation of these numbers as finite cardinals. $3 \cdot 4$, for example means the cardinal of a set, which is the set-theoretical union of three disjoint sets each with four elements.

The multiplication of positive rational numbers is introduced in close relationship to the interpretation of these numbers as ratios of a part to a whole. $2/3 \cdot 5/8$ for example means that a ratio which is obtained as follows: One forms 5/8 from a whole and forms 2/3 from that part. The ratio of this new part to the original whole is then $2/3 \cdot 5/8$. This is the classical interpretation of the multiplication as *forming portion-from*. In brief: *times* means *from*. The rule of multiplication of fractions (in brief: numerator times numerator, denominator times denominator) then can be proved.

The introductions of multiplication of the natural and the positive rational numbers are clearly application-oriented. The relationship under each other must remain unclear because it is a question of principally different things. The difficulties in the curricula arising there from are reduced by calculation exercises having fractions with a denominator of 1. The differences between *finite cardinals* and *ratios of a part to a whole* are downplayed.

The specified deficiencies both of Frege and of the curricula raise the question regarding construction of the number system having uniformity. This is in fact provided by the *comparational measurement theory*.

4 Didactical implications of the comparational measurement theory

4.1 Measuring as multiplicative or additive comparing

Frege, as mentioned already, had indeed raised the demand to include measuring into the definition of numbers. But it did not come down to a foundational analysis of measuring, so that Frege and the Neo-Fregeanism did not clarify how *quantities, measuring* and *numbers* are related together. Also, the *representational measurement theory* (Krantz et al., 1989) could not answer this question. However, the *comparational measurement theory* (cf. Griesel, 2005a), which was set up by the author as chairman of a working group in Deutsches Institut für Normung (DIN), does.

According to this theory measuring is primarily the multiplicative comparison of objects and values of quantities (ratio-scales). Numbers are the result of such comparisons. Secondly, an enlargement of this concept is measuring as an additive comparison of objects and values of interval-scales. The result of such an additive comparison is a quantity value of the comparison quantity, which appertains to every interval-scale.

4.2 Specification of the concepts of quantity (ratio-scale) and measuring

The following definition of the concept of quantity also involves the specification of the concept of measuring. Remember from Sect. 2.5 in this work that a quantity is a special mapping of a set of objects (line segments, surfaces, bodies, events, finite sets etc.) as the domain of the mapping and the set of values as the range of the mapping. This mapping is also called a scale.

4.2.1 Definition of a quantity

A mapping with the domain T and the range V is a *quantity (ratio-scale)* iff additionally an operation/with values in the set of real numbers ($\neq 0$) is defined in the range, which fulfils the following conditions:

$$(x/y)\cdot(y/z) = x/z \quad \text{for all } x,\ y,\ z \in V \tag{1}$$

$$x/y = 1 \Rightarrow x = y \quad \text{for all } x,\ y \in V \tag{2}$$

It has been proved that this definition is equivalent to the classical definition of a ratio-scale given by Stevens (1946) (Griesel, 1998).

The elements of the set T are called the *objects* of the quantity. They are embedded in reality. The elements of the set V are called the *values* of the quantity. The operation / is called *measuring* or *measuring quotient*. It is defined by a *measuring procedure*. $x/y = \lambda$ (read: x measured with y equals λ) means descriptively that y is contained in x lambda times. In this measurement x is compared multiplicatively with y. The result is the real number λ ($\neq 0$). The exact definition of the measuring procedure must use the objects of the quantity. Because the objects are interpreted as embedded in reality measuring is ontologically committed.

New in the definition of a quantity are especially the two conditions (1) and (2). New is not the perception of numbers as quotients of quantity values. The conditions specify what measuring in principle is. It can be proved that they are logical independent of each other.

It is spoken of the measurement x/y with the result λ. The two measurements x/y and y/z can be chained because y appears in both measurements (in the second and in the first place, respectively). The result of the chaining of x/y with y/z is x/z. Condition (1): $(x/y)\cdot(y/z) = x/z$ of the definition of the concept of quantity shows that if the measurements x/y and y/z are chained then the results are multiplied. Multiplication of real numbers and chaining of measurements are closely related to each other. This is an important fact for the following considerations.

It is also possible to define an addition + of quantity values. Then it is useful to introduce a zero-value 0 as the neutral element of addition. It is not reasonable here to go into detail.

4.3 Application-oriented introduction of numbers and multiplication

In the above definition of quantity are presupposed both the real numbers and the multiplication of these numbers. The definition shows the way how a uniform and application-oriented concept of number may be introduced in school. It can be stated:

Numbers are universal mental constructs, invented by human beings in order to be able to indicate intersub-

jectively the results of a measurement (in the sense of a multiplicative comparison of two values of a quantity).

In brief (Griesel, 2003): *Numbers are comparison results of measuring*.

This is a concept of number which considers all kinds of numbers – natural, positive rational, integer, real, irrational, even complex numbers – under a homogenous aspect. A fundamental change of the conceptual content which was found in Frege's considerations and which leads to difficulties in school curricula, is avoided.

The single kinds of numbers come from the application of the measuring procedure to different measuring systems (Griesel, 2006). The measuring system of the natural numbers is the class of finite sets. The measuring procedure consists of counting out a finite set, that is a multiplicative comparison of a finite set with sets of only one element. Natural numbers are in this sense primarily *counting numbers* (Bedürftig and Murowski, 2001).

The measuring systems of the positive rational numbers are the divisible quantity domains (in the sense of Kirsch, 1970) and of the positive real numbers the complete quantity domains.

The measuring systems of the positive and negative numbers are the Cartesian products of a quantity domain with a *one-dimensional direction characteristic*. The latter is a set with two elements (such as *to the right, to the left*; or *forward, backward*; or *up, down*; or *increase, decrease*; etc.). The measuring quotient is defined for such a set: $x/y = 1$ if $x = y$; $x/y = -1$ if $x \neq y$, where 1 and –1 are the two elements of the cyclic group of the order 2 (Griesel, 2003). Notice for this purpose that the multiplicative group of the positive and negative real numbers is the direct product of the group of order 2 with the multiplicative group of the positive real numbers.

As already mentioned it would have been more consistent if Frege had demanded not only the application of numbers for measuring but also that the application of multiplication and addition for measuring should be involved in their definition. Condition (1) in the definition of quantity $(x/y) \cdot (y/z) = x/z$ shows the way how this might be achieved. Multiplication of numbers has to be defined by chaining of measurements. A definition of multiplication on this basis could look as follows: $a \cdot b = c$, for which is valid: Let $a = x/y$ and $b = y/z$ then $c = x/z$.

It must still be shown that this definition is independent from the choice of x, y, z. It is in particular emphasized that this definition is valid for all kinds of numbers. It is a universal, application-oriented definition of multiplication with ontological commitments.

4.4 Specification of the concepts of interval-scale and additive measuring

In the following definition the concept of *interval-scale* is specified together with the concept of *additive measuring*. Similar to the concept of quantity an interval-scale is a mapping with a set of objects as the domain of this mapping and the set of values as the range of the mapping. To every interval-scale is associated a quantity Q.

4.5 Definition of an interval-scale

A mapping with the domain T, the range V and the associated quantity Q is an *interval-scale* iff additionally an operation – is defined in the range with values in the set of values of the quantity Q, which fulfils the following conditions:

$$(x - y) + (y - z) = x - z, \quad \text{for all } x, y, z \in V \qquad (1^*)$$

$$(x - y) = 0 \Rightarrow x = y, \quad \text{for all } x, y \in V \qquad (2^*)$$

It can be proved that this definition is equivalent to the classical definition of an interval-scale given by Stevens (1946).

The elements of T are called the *objects of the interval-scale*. The elements of the set V are called the *values of the interval-scale*. The operation – is called *additive measuring* or the *measuring difference*.

$x - y = a$ is read: *x additive measured with y equals a*. x is compared additively with y. The result a is a value of the quantity which is associated with the interval-scale. It is spoken of the additive measurement $x - y$ with the result a. The two additive measurements $x - y$ and $y - z$ can be chained because y is in both additive measurements (in the second and in the first place, respectively). The result is $x - z$. The chaining of $x - y$ with $y - z$ is $x - z$. Condition (1*) $(x - y) + (y - z) = x - z$ shows that if the additive measurements $(x - y)$ and $(y - z)$ are chained becoming $x - z$, then the results are added. Addition of quantity values and chaining of measurements are associated closely together. This is important for the following considerations about the addition.

4.6 Application-oriented introduction of addition

Addition can be introduced, correspondingly to multiplication, according to condition (1*) with the help of the chaining of additive measurements. However, it is an addition of quantity values rather than an addition of pure real numbers.

For all applications of addition to reality holds: *addition is an addition of quantity values.*

That was emphasized already at another place (Griesel, 2003).

The addition of pure numbers is without ontological commitments. An application-oriented introduction of addition therefore can only occur for quantity values.

The so-called *extensive quantities* have special importance for the addition of quantity values. These are quantities, for whose objects is defined a partial operation $\circ$, called *concate*, so that the values of the concatement $t_1 \circ t_2$ of the objects t_1 and t_2 is equal to the sum of the values of t_1 and t_2.

Examples of extensive quantities with high priority in school mathematics are *length, area, volume, mass, duration, finite cardinals, money*, and *angle*. However, the quantities *ratio of a part to a whole, temperature*, and *velocity* are not extensive quantities. Yet, they cannot be neglected so that the handling of addition in math education becomes complicated.

Also the quantity *change of the values of a quantity* is extensive. The concateness is the connection in series. This quantity is important for the handling of negative numbers in school mathematics. Further details will not be discussed here. (cf Postel, 2005).

4.7 Common basis for the application-oriented introduction of the numbers and their operations

The introduction of multiplication and addition as considered above agrees with the aspect indicated in Sect. 1 of this work. Both operations are introduced in an application-oriented fashion and indeed for all kinds of numbers in the same manner by chaining of measurements. Rules for the single kinds of numbers which are important for practical calculation can be derived from this introduction. Such rules are for instance for the positive rational numbers the rule *numerator times numerator, denominator times denominator* and for the negative numbers the rules of signs:

$$+ \textit{ times } + \textit{ is } +; \quad + \textit{ times } - \textit{ is } -;$$
$$- \textit{ times } + \textit{ is } -; \quad - \textit{ times } - \textit{ is } +.$$

There are corresponding rules for addition which however, are skipped here.

The program presented here of the application-oriented introduction of numbers and their operations might be understood as realization of Steiner's reform project for that part of school mathematics. At the present time it is not completely implemented in practice.

5 About the development of a corresponding curriculum

This reform program may be accomplished only after developing a corresponding practical curriculum and its implementation in school practice.

A step-by-step change of the present curricula is probably better than a complete revolution. Only thereby local weaknesses of the curriculum might be immediately eradicated without consequences for the total project.

The most difficult step is to separate the natural numbers as *numbers for counting out* from the idea of *finite cardinals* and the positive rational numbers from the idea of *ratios of a part to a whole* without neglecting these important quantities.

Didactical inventions and detailed analyses are still necessary for such a curriculum before it can be established in school practice.

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