(B) The subgroup $G$ is dense in $\mathfrak{R}$ (in the usual topology).

**Definition 2.** A function $f \in \mathfrak{F}$ is said to have principal period $p_0$ if $f$ is periodic, $G_f$ is of type $A$, and $p_0$ is the least positive element of $G_f$.

**Theorem 1.** If $f \in \mathfrak{F}$ is periodic, and if there exists a point $t_0 \in \mathfrak{R}$ such that $f$ is continuous at $t_0$, then either $f$ is a constant function or $f$ has a principal period.

**Proof.** Suppose that $f$ has no principal period, so that $G_f$ is dense in $\mathfrak{R}$. If $\epsilon > 0$, there exists $\delta > 0$ such that $|f(t) - f(t_0)| < \epsilon$ when $|t - t_0| < \delta$. If $t \in \mathfrak{R}$, we may express $t = t_0 + t' + t''$, where $t' \in G_f$ and $|t'| < \delta$, since $G_f$ is dense in $\mathfrak{R}$. Then

$$|f(t) - f(t_0)| = |f(t_0 + t' + t'') - f(t_0)| = |f(t_0 + t'') - f(t_0)| < \epsilon.$$ 

This shows that $f(t) = f(t_0)$, and $f$ is a constant function.

The "converse" of Theorem 1 is false. An example is the function $f$ given by $f(t) = 0$ ($t$ rational) and $f(t) = 2 + \cos 2\pi t$ ($t$ irrational). This function has principal period 1 but is everywhere discontinuous.

We now turn our attention to the functions of type B. Suppose that $G$ is a subgroup of $\mathfrak{R}$, and consider the factor group $\mathfrak{R}/G$. If $\hat{f}$ is a function from $\mathfrak{R}/G$ to $\mathfrak{R}$, we may use $\hat{f}$ to induce a function $f \in \mathfrak{F}$ whose period group is $G$ by defining $f(t) = \hat{f}(t + G)$. In particular, if $G$ is the subgroup $\mathfrak{R}$ of rationals, it is clear that the cardinality of $\mathfrak{R}/\mathfrak{R}$ is $\aleph_1$. In this case we may choose $f$ as a 1-1 mapping of $\mathfrak{R}/\mathfrak{R}$ onto $\mathfrak{R}$, and the resulting function $f$ is of type B and assumes uncountably many values. This construction will easily yield a periodic function whose graph is dense in the plane. As a final remark, we observe that there exist uncountable proper subgroups of $\mathfrak{R}$, and hence there exist non-constant periodic functions with uncountably many distinct periods.

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**MATHEMATICAL EDUCATION NOTES**

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**A DILEMMA IN DEFINITION**

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An unfortunate tendency of today's textbooks on calculus is to place the student in a quandary over the term "function." The definition of this term which had long commanded professional and academic acceptance is being supplanted in recent texts by substitute versions which—although heralded as improvements in precision—fail notably on grounds of consistency. The re-
result is an ambiguity which even the most careful mathematical writers are unable to avoid. There is danger that the new definitions are producing an irreconcilable conflict with a meaning long established in the literature of science and engineering. If this tendency continues, a student’s first course in calculus seems destined to become a bewildering encounter with inconsistent language and overly elaborate symbolism.

In order to discuss the problem in specific terms, it will be helpful to state the conflicting definitions and assign corresponding marginal references. The definition of function that was widely accepted until about ten years ago appeared in substantially the following terms:

(A) *A function is a variable so related to another variable that to each value of the latter there corresponds uniquely a value of the former.*

In contrast, the definition tending to prevail in the most recently published calculus texts would be as follows:

(B) *A function is a set of ordered pairs, no two of which have the same first component.*

Still another view regards a function as a rule of correspondence. There are many ways of stating this, but the following paraphrase is sufficiently explicit for later reference:

(C) *Given a set of ordered pairs such that no two have the same first component, the rule of correspondence which associates each second component with the corresponding first is called a function.*

Certain variants of these definitions could be added to the list, but it will suffice to focus attention on (A), (B) and (C).

Now, under all three definitions the symbol \( f(x) \) is customarily used to denote what is called the *value* of the function. Further, if we let \( (x, y) \) denote an ordered pair in the sense of (B) and (C), then under all three definitions it is customary also to consider \( y \) as the value of the function. Thus \( y \) and \( f(x) \) are given the same meaning, so that in all three systems it is acceptable to write \( y=f(x) \). Regardless of which definition is chosen, therefore, the symbolism prevalent in science and engineering may be used whenever one’s only concern is the function-value corresponding to a selected argument-value. Accordingly, if the only purpose of mathematical education were to prepare the student for stereotyped problems, one might say that the fundamental conflict of language between (A), (B) and (C) is of small consequence. But to take this attitude would be to revive the cookbook approach that the recent reforms have been intended to remedy.

If each of the definitions (A), (B) and (C) be reduced to the essential single word, we find that (A) calls a function a variable while (B) calls it a set, and (C) calls it a rule. These three meanings are not logically equivalent; and beyond that, (B) and (C) are at variance with the meaning of function most widely accepted in applications of calculus to science and engineering. By this I do not mean to belittle the concepts defined in (B) and (C); I am fully aware that these are important objects of interest, and that each deserves a standard term in
mathematical literature. My only point is that neither of these should be referred to as a function, since this terminology conflicts not only with a previously accepted mathematical definition but also with a fundamental meaning established in the English language by prolonged good usage.

As to what terms I would propose instead for (B) and (C), well suited words are available, but there is little point in discussing these unless teachers first agree that the overthrow of definition (A) is neither necessary nor desirable. Definition (A) is fully adapted to a precise treatment of calculus, and the usage implied by that definition is so deeply imbedded in the literature that it is not likely to vanish in our lifetime. Agreement on these points would make it clear that the term “function” cannot reasonably be applied to (B) or (C), if calculus is to be kept free of ambiguity. The problem would then reduce primarily to that of choosing an acceptable name for the concept denoted by the symbol $f$ when used separately from $f(x)$. A reasonable solution to this problem was suggested as far back as 1953 by J. Barkley Rosser, whom I shall now quote. In his admirable *Logic for Mathematicians*, (McGraw-Hill, 1953), Rosser offers on page 309 the following suggestion:

Actually, a perfectly good name for the notion $f$ is available, namely “transformation.” In algebraic geometry, a careful distinction is usually made between a “transformation” $f$ and a “general value” $f(x)$ of the transformation. Thus, one way out of our impasse would be always to refer to $f$ as a transformation and to reserve the term “function” to refer to $f(x)$.

This suggestion in Rosser’s book occurs in the course of an illuminating discussion of the dilemma which is the subject of the present article. Pointing out that $f$ and $f(x)$ are different and that it is confusing to apply the term “function” to both, he continues as follows:

One should definitely decide to call one by the name “function” and then devise a new name for the other. The present trend in higher mathematics is to reserve the name “function” for $f$, but even those who advocate this are usually inconsistent in their use of the word “function.”

Rosser proceeds from this point with a description of certain difficulties raised by calling $f(x)$ a function, but after close study I am persuaded that the problem is a minor one whose solution requires no more formidable remedy than careful English. Moreover, Rosser shows convincingly that symbolic logic provides adequate machinery for considering $f(x)$ a function and treating it in that sense with complete precision.

After describing various alternatives, Rosser reaches the point where a decision is necessary in order to proceed with the treatment of functions in his own pages. For this purpose he accepts the emerging practice of calling $f$ the function, and in order to preserve consistency he is careful thereafter to refer to $f(x)$ as the function value. He points out that the procedure thus adopted is not highly satisfactory, but seems most nearly in accord with the present trend of mathematical thought.

In order to illustrate the growing dilemma I shall cite a few examples of more recent date and at the same time try to clarify the nature of the problem. An
important source of trouble is doubt in the minds of certain writers that the concept of a variable can be made precise. A person who does not know what a variable is would, of course, not understand definition (A). Some authors try to avoid the variable concept altogether, a case in point being the writer of a prominent introductory treatment of functions published in 1960. At one stage he remarks, "For the purposes of this book it is unnecessary to attempt to define 'a variable,' and we shall neither define nor use this word." Later on, however, he freely uses the verbs "increase" and "decrease," and he does not hesitate to state that certain functions are increasing or decreasing. Thus he has used the idea of a variable while avoiding the word. This in itself might be regarded as not serious, except for the fact that he had previously defined a function as a set of ordered pairs. Thus, he seems to be saying that a set of ordered pairs is increasing or decreasing.

I do not believe, of course, that he meant this, nor would an experienced mathematician insist on this literal interpretation. His book is intended for readers who have completed at least a course in calculus. Nevertheless, even at the level for which he was writing, the need to discuss functional behavior in intelligible language forced him to revert to the forms of speech which treat a function as a variable. Moreover, it is clear that he was aware of the language problem, for shortly afterward he comments that the function whose value at x is log x would usually be called "the function log x." Proceeding then to the function whose value at x is log(sin x), he states:

Here we are forced into a certain amount of awkwardness if we are to avoid the possibly misleading "the function log (sin x)." The same problem arises when we want to talk about functions that are so simple that they have no generally accepted names.

In view of the uncontradicted conflict of language in this treatment, a student who reads it would be left in serious doubt as to whether such entities as \(x^2\), sin x, log(sin x), and so on, may properly be referred to as functions.

Several modern authors have been frank to acknowledge the dilemma, and after defining a function in accordance with the new vogue, they revert openly to the variable concept in order to provide intelligible discussion. Thus the preface of a certain text on calculus published in 1964 includes the following confession:

The notation \(f(x)\) is defined with complete propriety as the image of the function \(f\) for a given pre-image \(x\): that is, \(y=f(x) \iff (x, y) \in f\). But to avoid too great a clash with tradition and with the bulk of mathematical literature, we later slip into the (bad?) habit of calling \(f(x)\) a function.

The text then proceeds, commendably in my opinion, to refer to such entities as \(x^2\), \(x^3\), sin x, and so on, as functions.

There are other authors who have gone to great lengths to avoid such overt inconsistency, but the inevitable consequence is excessive symbolism and circumlocution. This is well illustrated by a textbook on calculus written by two prominent authors and published in 1962. These writers define a function to be a set of ordered pairs no two of which have the same first component, and as a
typical symbol they employ the letter $F$. Their preface introduces the following conventions for distinguishing between $F$ and $F(x)$:

If $F$ denotes a function, and if $(x, y) \subseteq F$, we call $y$ the *correspondent* of $x$ under $F$, and we denote this correspondent by $F(x)$. We most commonly specify a function by writing

$$F = \{(x, y) \mid y = F(x)\},$$

and if $(a, F(a)) \subseteq F$, we call $F(a)$ the value of $F(x)$ at $a$.

This device enables its authors to avoid the logical inconsistency of first saying that a function $F$ is a set, and then saying that $F(a)$ denotes a value of the function (literally, a "value of the set $F$"). Instead, they say that $F(a)$ is a value of the *correspondent*, thus ascribing to the latter concept the intrinsic character of a variable.

While I admire the sincerity with which these authors have insisted on consistency, I feel nevertheless that they have demanded of the student too high a price. The student who uses this book will be dealing, not only with functions, but also with correspondents; and not only with derivatives of functions, but also with derivatives of correspondents; and not only with antiderivatives of functions, but also with antiderivatives of correspondents. As for notation, in cases where educated men have long felt that the display

$$\sin x$$

denotes a function, the student of this 1962 text will be required instead to denote the function by the display

$$\sin = \{(x, y) \mid y = \sin x\}.$$

Since a function is a set of ordered pairs, this student may not say that a function is capable of increasing or decreasing or having an extreme value.

Now, without discussing the mathematical merits of these dual conventions, I must view them with regret from the standpoint of good teaching. Calculus can be made simpler than this without impairing either its rigor or its precision. Moreover, a student who masters these conventions may find himself in a strange and conflicting world of applications. His physics teacher will probably persist in calling $\sin x$ a function, and may also insist that the distance traveled by a falling stone is a function of the time elapsed since the stone was released. The latter observation may cause the student to ponder how a distance can possibly be a set of ordered pairs. Such conflicts between mathematical fiat and the established terms of science and engineering are not likely to advance the cause of mathematical education. Rather, it seems to me, calculus is in danger of becoming esoteric, and those of us who teach it are in danger of being considered a *cult* if we allow eagerness for modernization to do violence to correct usage and accepted terminology.

The mathematician may treat usage with scorn, of course, as did Humpty Dumpty in his imperious reply to Alice: "When I use a word, it means just what
I choose it to mean—neither more nor less.” But I am not convinced that mathematicians in the aggregate wish to carry this privilege to extremes. At the highest levels of research it may be that a writer can expect tolerance from his colleagues if his preoccupation is so intense that his language becomes a law unto itself. Nevertheless, a book that flouts the prevailing language of science and engineering does not constitute good text material. The student will find that functions are not exclusively in the province of mathematicians, but pervade also the business of the laboratory and the market place.

The fact of the 1960’s is that mathematicians, physicists, engineers, and increasing numbers of people in the behavioral and management sciences, are accustomed to making certain types of assertions with regard to functions, such as the following:

“This function always increases.”
“This function changes always at the same rate.”
“This function takes only positive values.”

Under these circumstances, a definition which states what a function is should be such that the essential predicate nominative may be substituted for the word “function” in each of the above sentences. In all three cases the term “variable” fits logically, whereas the term “set of ordered pairs” does not, nor does the term “correspondence,” or “rule of correspondence,” or “mapping.” Either we must adhere to the concept that a function is a variable, or else we face the long and difficult battle of uprooting established forms of speech. In such a battle the first casualty, in my opinion, will be the reputation of calculus as a reasonable academic discipline.

As for those who doubt that the concept of a variable can be made precise, I suggest that teachers should explore this problem in a fundamental way, rather than accept a dictum. It is widely agreed, I assume, that a major purpose of calculus is to create effective models of physical situations. If it is now becoming a status-symbol to profess doubt as to what a variable is, nevertheless nature continues to go about the business of variation with carefree abandon. Antelopes run, birds fly, rockets accelerate, apples fall, nuclear bombs explode, and in general the universe seems in a state of incessant change. The invention of calculus resulted from an effort to model certain variations of nature, notably the variations of distance and velocity in a system where a body in orbit is subject to a gravitational force directed toward a center of mass.

Accordingly a characteristic task of calculus is to create a model of something that varies, and any mathematical entity that may serve this purpose (even if not so used) is properly characterized by the term “variable.” Moreover, it is not difficult to define this term in a way that is compatible with the language of sets and at the same time provides a foundation for definition (A).

The language which calls \( f(x) \) a function and considers it a variable proves indispensable whenever mathematicians are confronted with practical affairs, and this is noticeable at the most authoritative levels. Thus in the Report of the
Cambridge Conference on School Mathematics (Educational Services Incorporated, 1963), one finds on page 59 that a certain differential equation with a side condition "determines a unique function \( f(x) \)." On page 61 the same report suggests the use of Taylor's series to obtain the series for "the standard functions such as \( \log(1+x) \) and \( (1+x)^n \)." In 1964 the William Lowell Putnam examination set the stage for a problem with the command "Let \( f(x) \) be a real continuous function defined for all \( x \)," and in another problem the competitor was required to find "all continuous functions \( f(x) \), for \( 0 \leq x \leq 1 \), such that . . . ." If this is to be the language of professionals, why should we exclude it from the classroom?

When viewed in retrospect, the campaign to overthrow definition (A) seems an exercise in futility. The fact remains that \( f(x) \), whatever one may call it, stands indispensable as the center of interest in calculus. It still has the property that educated people attribute to a variable. It is still true that, under suitable specifications, every assignment of a value to \( x \) determines uniquely a value of \( f(x) \). It is still true that the English language regards a function as "any quality, trait, or fact so related to another that it is dependent upon and varies with that other." It is still true that usage, and history, and etymology, as well as the vast weight of established literature, regard \( f(x) \) as a function.

Under these conditions, if the teachers and authors of calculus are supposed not to call \( f(x) \) a function, then we are confronted with three (shall I say needlessly self-imposed?) problems. First, it will be necessary to find a generally accepted new name for \( f(x) \); second, the established literature of science and engineering will have to be readjusted to a fundamental change in terminology; and third, we shall have to agree on what it is that shall finally be called a function. With regard to this third problem, those who would deprive \( f(x) \) of its long-enduring name are not in agreement as to where to bestow the prize. Some would give the title to a set of ordered pairs, others to a correspondence, others to a rule of correspondence, others to a mapping. But none of these candidates (all admittedly important concepts) can reasonably be called a function, if the language we speak and derive from established literature is to play its proper role in mathematical education.

**A STUDY OF WAYS OF HANDLING LARGE CLASSES IN FRESHMAN MATHEMATICS**

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Increased enrollments along with a desire to maintain personal contact between student and instructor make it highly desirable to explore ways of handling large classes.

The problem. Research indicates that learning in large lecture classes is not generally inferior to that in smaller lecture classes if one uses traditional achievement tests as a criterion, but experiments suggest that fewer students raise questions or interpose comments in large classes than in small. There may not