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Who Gave You the Cauchy–Weierstrass Tale? The Dual History of Rigorous Calculus

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Abstract Cauchy's contribution to the foundations of analysis is often viewed through the lens of developments that occurred some decades later, namely the formalisation of analysis on the basis of the epsilon-delta doctrine in the context of an Archimedean continuum. What does one see if one refrains from viewing Cauchy as if he had read Weierstrass already? One sees, with Felix Klein, a parallel thread for the development of analysis, in the context of an infinitesimal-enriched continuum. One sees, with Emile Borel, the seeds of the theory of rates of growth of functions as developed by Paul du Bois-Reymond. One sees, with E. G. Björling, an infinitesimal definition of the criterion of uniform convergence. Cauchy's foundational stance is hereby reconsidered.

Keywords Archimedean axiom · Bernoulli · Cauchy · Continuity · Continuum · du Bois-Reymond · Epsilontics · Felix Klein · Hyperreals · Infinitesimal · Stolz · Sum theorem · Transfer principle · Ultraproduct · Weierstrass

1 Introduction

When the second-named author first came across a recent book by Anderson *et al.* entitled *Who Gave You the Epsilon? And Other Tales of Mathematical History* (Anderson et al. 2009), he momentarily entertained a faint glimmer of hope. The book draws its title from an older essay, entitled *Who Gave You the Epsilon? Cauchy and the origins of rigorous calculus* (Grabiner 1983). The faint hope was that the book would approach the thesis implied

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by the title of the older essay, in a critical spirit, namely, as an ahistorical[1] *tale* in need of re-examination. Anderson *et al* not having undertaken the latter task, such an attempt is made here.

Cauchy based his definitions of both limits and infinitesimals on the concept of a variable quantity.[2] The variable quantities used in his *Cours d'Analyse* in 1821 are generally understood by scholars as being (discrete) sequences of values. Cauchy wrote that a variable quantity tending to zero becomes infinitely small. How do his null sequences become infinitesimals? In 1829, Cauchy developed a detailed theory of infinitesimals of arbitrary order (not necessarily integer), based on rates of growth of functions. How does his theory connect with the work of later authors? In 1902, E. Borel elaborated on du Bois-Reymond's theory of rates of growth, and outlined a general "theory of increase" of functions, as a way of implementing an infinitesimal-enriched continuum. Borel traced the lineage of such ideas to Cauchy's text. We examine several views of Cauchy's foundational contribution in analysis.

Cauchy's foundational stance has been the subject of an ongoing controversy. The continuing relevance of Cauchy's foundational stance stems from the fact that Cauchy developed some surprisingly modern mathematics using infinitesimals. The following four items deserve to be mentioned:

(1) *Cauchy's proof of the binomial formula (series) for arbitrary exponents*. The proof exploits infinitesimals. Laugwitz (1987, p. 266) argues that this is the first correct proof of the formula.[3]

(2) *Cauchy's use of the Dirac delta function*. Over a century before Dirac, Cauchy exploited "delta functions" to solve problems in Fourier analysis and in the evaluation of singular integrals. Such functions were defined in terms of an infinitesimal parameter, see Freudenthal (1971), Laugwitz (1989, p. 219, 1992).

(3) *Cauchy's definition of continuity*. Cauchy defined continuity of a function $y = f(x)$ as follows: *an infinitesimal change α of the independent variable x always produces an infinitesimal change $f(x+\alpha) - f(x)$ of the dependent variable y* (Cauchy 1821, p. 34). Nearly half a century later, Weierstrass reconstructed Cauchy's infinitesimal definition in the following terms: for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every real α, if $|\alpha| < \delta$ then $|f(x+\alpha) - f(x)| < \epsilon$. Many historians have sought to interpret Cauchy's definition as a proto-Weierstrassian definition of continuity in terms of limits.[4]

(4) *Cauchy's "sum theorem"*. This result asserts the convergence to a *continuous* function, of a series of continuous functions under a suitable condition of convergence. The sum theorem has been the subject of a historical controversy, ever since Robinson (1966, pp. 271–273) proposed a novel reading of the sum theorem that would make

[1] See Grattan-Guinness's comment on ahistory in the main text around footnote 65.

[2] Gilain's claims to the effect that "Cauchy définissait le concept d'infiniment petit à l'aide du concept de limite, qui avait le premier rôle" (Gilain 1989, footnote 67) are both in error; see Subsect. 3.1 below. Already in 1973, Hourya Benis Sinaceur wrote: "on dit trop rapidement que c'est Cauchy qui a introduit la 'méthode des limites' entendant par là, plus ou moins vaguement, l'emploi systématique de l'*épsilonisation*" (Sinaceur 1973, p. 108), and pointed out that Cauchy's definition of limit resembles, not that of Weierstrass, but rather that of Lacroix dating from 1810 (Sinaceur 1973, p. 109).

[3] Except for Bolzano's proof in 1816, see Laugwitz (1997, p. 657).

[4] Thus, Smithies (1986, p. 53, footnote 20) cites the *page* in Cauchy's book where Cauchy gave the infinitesimal definition, but goes on to claim that the concept of *limit* was Cauchy's "essential basis" for his concept of continuity (Smithies 1986, p. 58). Smithies looked in Cauchy, saw the infinitesimal definition, and went on to write in his paper that he saw limits (see Sect. 3.1 for an analysis of a similar misconception in Gilain). Such automated translation has been common at least since Boyer (1949, p. 277).

> it correct. The controversy hinges on the question whether the convergence condition was meant by Cauchy to hold at the points of an Archimedean continuum, or at the points of a Bernoullian continuum, namely, an infinitesimal-enriched continuum (see Sect. 2.4). Lakatos presented a paper[5] in 1966 where he argues that the 1821 result is correct as stated.[6] Laugwitz concurs, see e.g. his 1989 text (Laugwitz 1989). Post-Weierstrassian historians tend both (a) to reject Cauchy's infinitesimals, claiming they are merely shorthand for limits, and (b) to claim that his 1821 sum theorem was false. The 1821 formulation of the sum theorem may in the end be too ambiguous to know what Cauchy's intention was at the time, if Cauchy himself knew.[7]

The four examples given above hopefully illustrate the interest in understanding the nature of Cauchy's foundational contribution.

In Sect. 2, we provide a re-appraisal of Cauchy's foundational stance, including Cauchy's theory of orders of infinitesimals based on orders of growth of functions (Sect. 2.1); how Cauchy null sequences become infinitesimals (Sect. 2.2); a detailed textual study of the related portions of the *Cours d'Analyse* (Sect. 2.3); an analysis of Cauchy's key term *toujours*, strengthening the hypothesis of his sum theorem (Sect. 2.4); an examination of some common misconceptions in the Cauchy literature (Sect. 2.5).

In Sect. 3, we analyze the views of some modern authors. In Sect. 4, we provide a timeline of the development of theories of infinitesimals based on orders of growth of functions, from Cauchy through Paul du Bois-Reymond to modern times. Some conclusions and outlook for the future may be found in Sect. 5. Appendix A outlines the relevant mathematical material on the rival continua, Archimedean and Bernoullian.

2 A Reappraisal of Cauchy's Foundational Stance

2.1 Cauchy's Theory of Orders of Infinitesimals

Cauchy's 1821 *Cours d'Analyse* (Cauchy 1821) presented only a theory of infinitesimals of polynomial rate of growth as compared to a given "base" infinitesimal α.

The shortcoming of such a theory is its limited flexibility. Since Cauchy only considers infinitesimals behaving as polynomials of a fixed infinitesimal, called the "base" infinitesimal in 1823 (Cauchy 1823), his framework imposes obvious limitations on what can be done with such infinitesimals. Thus, one typically can't extract the square root of such a "polynomial" infinitesimal.

What is remarkable is that Cauchy did develop a theory to overcome this shortcoming. Cauchy's theory of infinitesimals of arbitrary order (not necessarily integer) is noted by Borel (1902, pp. 35–36).

In 1823, and in more detail in 1829, Cauchy develops a more flexible theory, where an infinitesimal is represented by an arbitrary *function* (rather than merely a polynomial) of a base infinitesimal, denoted "i", see Chapter 6 in Cauchy (1829). The title of the chapter is significant. Indeed, the title refers to functions as *representing* the infinitesimals; more precisely, *"fonctions qui représentent des quantités infiniment petites"*. Here is what Cauchy has to say in 1829:

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[5] The paper was published posthumously in 1978 and edited by Cleave, see (Lakatos 1978).

[6] Lakatos thereby reversed his position as presented in his *Proofs and Refutations* (Lakatos 1976).

[7] See more details in Sect. 2.4.

> Designons par a un nombre constant, rationnel ou irrationnel; par i une quantité infiniment petite, et par r un nombre variable. Dans le systeme de quantités infiniment petites dont i sera la base, une fonction de i representée par $f(i)$ sera un infiniment petit de l'ordre a, si la limite du rapport $f(i)/i^r$ est nulle pour toutes les valeurs de r plus petites que a, et infinie pour toutes les valeurs de r plus grandes que a (Cauchy, 1829, p. 281).

Laugwitz (1987, p. 271) explains this to mean that the order a of the infinitesimal $f(i)$ is the uniquely determined real number (possibly $+\infty$, as with the function e^{-1/t^2}) such that $f(i)/i^r$ is infinitesimal for $r < a$ and infinitely large for $r > a$.

Laugwitz (1987, p. 272) notes that Cauchy provides an example of functions defined on positive reals that represent infinitesimals of orders ∞ and 0, namely

$$e^{-1/i} \quad \text{and} \quad \frac{1}{\log i}$$

(see Cauchy 1829, pp. 326–327).

The development of non-Archimedean systems based on orders of growth was pursued in earnest at the end of the 19th century by such authors as Stolz, du Bois-Reymond, Levi-Civita, and Borel.[8] Such systems have an antecedent in Cauchy's theory of infinitesimals as developed in his texts dating from 1823 and 1829. Thus, in 1902, Borel (1902, pp. 35–36) elaborated on du Bois-Reymond's theory of rates of growth, and outlined a general "theory of increase" of functions, as a way of implementing an infinitesimal-enriched continuum. Borel traced the lineage of such ideas to an 1829 text of Cauchy's on the rates of growth of functions, see Fisher (1979, p. 144) for details. In 1966, A. Robinson pointed out that

> Following Cauchy's idea that an infinitely small or infinitely large quantity is associated with the behavior of a function $f(x)$, as x tends to a finite value or to infinity, du Bois-Reymond produced an elaborate theory of orders of magnitude for the asymptotic behavior of functions …Stolz tried to develop also a theory of arithmetical operations for such entities (Robinson 1966, pp. 277–278).

Robinson traces the chain of influences further, in the following terms:

> It seems likely that Skolem's idea to represent infinitely large natural numbers by number-theoretic functions which tend to infinity (Skolem 1934),[9] also is related to the earlier ideas of Cauchy and du Bois-Reymond (Robinson 1966, p. 278).

Cauchy's approach is by no means a variant of Robinson's approach.[10] Similarly, the notion of Cauchy as a pre-Weierstrassian who allegedly "tried to dislodge infinitesimals from analysis" is just as dubious. Taken to its logical conclusion, the dogma of Cauchy as a pre-Weierstrassian can assume comical proportions. Thus, the fashionable Stephen Hawking comments that Cauchy

> was particularly concerned to banish infinitesimals (Hawking 2007, p. 639),

[8] See Ehrlich (2006) for a more detailed discussion, as well as Sect. 4 below.

[9] The reference is to Skolem's 1934 work (Skolem 1934). The evolution of modern infinitesimals is traced further in the main text in Sect. 4 following footnote 71.

[10] More specifically, Cauchy was not in the possession of the mathematical tools required to either formulate or justify the ultrapower construction, requiring as it does a set-theoretic framework (dating from the end of the nineteenth century) together with the existence of ultrafilters (not proved until 1930 by Tarski 1930), see Appendix A.

yet *on the very same page* 639, Hawking quotes Cauchy's *infinitesimal* definition of continuity in the following terms:

> the function $f(x)$ remains continuous with respect to x between the given bounds, if, between these bounds, an infinitely small increment in the variable always produces an infinitely small increment in the function itself (Hawking 2007, p. 639).

Did Cauchy "banish" infinitesimals? Using infinitely small increments is an odd way of doing so. Similarly, historian J. Gray lists *continuity* among concepts Cauchy allegedly defined

> using careful, if not altogether unambiguous, **limiting** arguments (Gray 2008, p. 62) [emphasis added–authors],

whereas in point of fact, *limits* appear in Cauchy's definition only in the sense of the *endpoints* of the domain of definition. Analogous misconceptions in Gilain are analyzed in Sect. 3.1.[11] Similar post-Weierstrassian sentiments were expressed by editor Michel Blay's referee for the periodical *Revue d'histoire des sciences* where the present article was submitted for publication in 2010, and rejected based on the following assessment by referee 1:[12]

> Our author interprets A. Cauchy's approach as a formation of the idea of an infinitely small—a variant of the approach which was developed in the XXth century in the framework of the nonstandard analysis (a hyperreal version of E. Hewitt, J. Los, A. Robinson).

The referee's summary is an inaccurate description of the text refereed. Cauchy's approach is not a variant of Robinson's, and was never claimed to be in this text. Based on such a strawman version of the article's conception, the referee came to the following conclusion:

> From my point of view the author's arguments to support this conception are quite unconvincing.

Indeed, we find such such a strawman conception unconvincing, but the conception was the referee's, not the authors'. The referee concluded as follows:

> The fact that the actual infinitesimals lived somewhere in the consciousness of A. Cauchy (as in many another mathematicians of XIXth–XXth centuries as, for example, N.N. Luzin) does not abolish his (and theirs) constant aspiration to dislodge them in the subconsciousness and to found the calculus on the theory of limit.[13]

Did Cauchy have a "constant aspiration to dislodge infinitesimals in the subconsciousness and to found the calculus on the theory of limit"? Certainly not. Felix Klein knew better: fifty years before Robinson, Klein clearly realized the potential of the infinitesimal approach to the foundations. Having outlined the developments in real analysis associated with Weierstrass and his followers, Klein pointed out that

> The scientific mathematics of today is built upon the series of developments which we have been outlining. But an essentially different conception of infinitesimal calculus has been running parallel with this [conception] through the centuries (Klein 1932, pp. 214).

[11] See Katz and Katz (2011a,b) for further discussion of the post-Weierstrassian bias in Cauchy scholarship.

[12] The pdf of the version submitted to Blay, as well as the two referee reports, may be found at http://www.u.cs.biu.ac.il/~katzmik/straw.html.

[13] Opposing infinitesimals and limits is in itself a conceptual error; see footnote 33.

Such a different conception, according to Klein, "harks back to old metaphysical speculations concerning the structure of the continuum according to which this was made up of […] infinitely small parts" (Klein 1932, p. 214).[14]

Cauchy did not aspire to dislodge infinitesimals from analysis; on the contrary, he used them with increasing frequency in his work, including his 1853 article (Cauchy 1853) where he relied on infinitesimals to express the property of uniform convergence, as we analyze in Sect. 2.4.

2.2 How Does a Null Sequence Become a Cauchy Infinitesimal?

The nature of Cauchy's infinitesimals has been the subject of an ongoing debate for a number of decades. Post-Weierstrassian historians tend to dismiss Cauchy's *infiniment petits* as merely a linguistic device masking Cauchy's use of the limit concept. From this perspective, Cauchy's infinitesimals are alleged to represent an early anticipation of the more rigorous methods developed in the second half of the 19th century, namely epsilontic analysis. Sinaceur (1973) presented a critical analysis of the proto-Weierstrassian approach to Cauchy in 1973, pointing out in particular that Cauchy's notion of limit is for the most part a kinetic one rather than an epsilontic one.[15] Other historians have taken Cauchy's infinitesimals at face value, see e.g., Lakatos (1978), Laugwitz (1989). The studies in the past decade include Sad et al. (2001), Bråting (2007), and Katz and Katz (2011b).

To what extent did Cauchy intend the process that he described as a null sequence "becoming" an infinitesimal, to involve some kind of a collapsing?

Cauchy did not have access to the modern set theoretic mentality (currently dominant in the area of mathematics), where equivalence relation and quotient space constructions are taken for granted. One can still ponder the following question: to what extent may Cauchy have anticipated such collapsing phenomena?

Part of the difficulty in answering such a question is Cauchy's hands-on approach to foundations. Cauchy was less interested in foundational issues than, say, Bolzano.[16] To Cauchy, getting your sedan out of the garage was the only justification for shoveling away the snow that blocks the garage door. Once the door open, Cauchy is in top gear within seconds, solving problems and producing results. To him, infinitesimals were the asphalt under the snow, not the snow itself. Bolzano wanted to shovel away *all* the snow from the road. Half a century later, the triumvirate[17] shovel ripped out the asphalt together with the snow, intent on consigning the infinitesimal to the dustbin of history.

For these reasons, it is not easy to gauge Cauchy's foundational stance precisely. For instance, is there evidence that he felt that two null sequences that coincide except for a finite

[14] Klein also formulated a criterion of what it would take for a theory of infinitesimals to be successful. Namely, one must be able to prove a mean value theorem for arbitrary intervals, including infinitesimal ones. In 1928, Fraenkel (1946, pp. 116–117) formulated a similar requirement in terms of the mean value theorem. Such a Klein-Fraenkel criterion is satisfied by the Hewitt-Łoś-Robinson theory by the transfer principle, see Appendix A.

[15] Pourciau (2001) argues that Newton possessed a clear kinetic conception of limit similar to Cauchy's, and cites Newton's lucid statement to the effect that "Those ultimate ratios…are not actually ratios of ultimate quantities, but limits…which they can approach so closely that their difference is less than any given quantity…" see Newton (1946, p. 39, 1999, p. 442). The same point, and the same passage from Newton, appeared a century earlier in Russell (1903, item 316, pp. 338–339). See also footnote 33.

[16] Thus, Hourya Benis Sinaceur writes: "Bolzano est avant tout préoccupé de rigueur théorique, d'où le soucit constant de démonstrations formelles […] Cauchy, au contraire, donne rarement des démonstrations en forme […] Son exposé a davantage des qualités de *synthèse* que de rigueur formelle" (Sinaceur 1973, p. 102).

[17] Boyer (1949, p. 298) refers to Cantor, Dedekind, and Weierstrass as "the great triumvirate".

number of terms, would "generate" the same infinitesimal?[18] Support for this idea comes from several sources, including a very unlikely one, namely Felscher's essay (Felscher 2000) in *The American Mathematical Monthly* from 2000, where he attacks both Laugwitz's interpretation and the idea that infinitesimals play a foundational role in Cauchy.[19]

In his zeal to do away with Cauchy's infinitesimals, Felscher seeks to describe them in terms of the modern terminology of *germs*. Here two sequences are in the same germ if they agree at infinity, i.e., for all sufficiently large values of the index n of the sequence $\langle u_n : n \in \mathbb{N}\rangle$, that is, are equal almost everywhere.[20] Felscher's maneuver successfully eliminates the term "infinitesimal" from the picture, but has the effect of undermining Felscher's own thesis, by lending support to the presence of such "collapsing" in Cauchy.

Indeed, the idea of reading germs of sequences into Cauchy is precisely Laugwitz's thesis in Laugwitz (1987, p. 272). Germs of sequences are also the basis of Laugwitz's Ω-calculus (Schmieden and Laugwitz 1958), a ring constructed using a Fréchet filter.[21]

In an editorial footnote to Lakatos's essay, J. Cleave outlines a non-Archimedean system developed by Chwistek (1948), involving a quotient by a Fréchet filter (similarly to the Schmieden–Laugwitz construction). Cleave then concludes that the relation of such infinitesimals to Cauchy's is "obvious".[22] To us it appears that the said relation requires additional argument.

Cauchy's published work contains evidence that he intuitively sensed a "collapsing" involved in the passage from a null sequence to an infinitesimal. Thus, the second chapter of the *Cours d'Analyse* (Cauchy 1821) of 1821 contains a series of theorems (eight of them) whose main purpose appears to be to emphasize the importance of the *asymptotic* behavior of the sequence (i.e., as the index tends to infinity). This thread is pursued in more detail in Sect. 2.3. Furthermore, he refers to his infinitesimals as "quantities", a term he uses in the context of an ordered number system, as opposed to the complex numbers which are always "expressions" but never "quantities". In fact, the recent English translation (Bradley and Sandifer 2009) of the *Cours d'Analyse* erroneously translates one of Cauchy's complex "expressions" as a "quantity", and the reviewer for *Zentralblatt* dutifully notes this error.[23]

A second aspect of collapsing is the mental transformation of the process of "tending to zero" into a concept/noun (as a null sequence is transformed into an infinitesimal), thought

[18] Note that the term "generate" was used by Bråting (2007) to describe the passage from sequence to infinitesimal in Cauchy.

[19] Felscher's text is examined in more detail in Sect. 3.2.

[20] Equality "almost everywhere" is a twentieth century concept; Felscher seems to suggest that Cauchy thought of the relation between his variable quantities and his infinitesimals in a way that would be later described as equality almost everywhere.

[21] In the Schmieden–Laugwitz construction, a Fréchet filter is used where an ultrafilter would be used in a hyperreal construction; see Appendix A.

[22] Cleave writes in his footnote 32*: "A construction of non-standard analysis is given in Chwistek (1948) which is derived from a paper published in 1926. It is basically the reduced power $\mathbb{R}^{\mathbb{N}}/F$ where F is the Fréchet filter on the natural numbers (the collection of cofinite sets of natural numbers) (see Frayne et al. (1962-3)) [...] This particular construction is not an elementary extension of $\mathbb{R}$ but there are sufficiently powerful transfer properties to enable some non-standard analysis to be performed. It may be observed that the elements of $\mathbb{R}^{\mathbb{N}}/F$ are equivalence classes of sequences of reals, two sequences $s_1, s_2 \dots$ and $t_1, t_2, \dots$ being counted equal if for some n, $s_m = t_m$ for all $m \geq n$. The relation of these classes to Cauchy's variables is obvious" (Lakatos 1978, p. 160).

[23] The reviewer, Reinhard Siegmund-Schultze, was far more sensitive to Cauchy's infinitesimals on this occasion than in an earlier instance, occasioned by a review of Felscher's text; see footnote 56.

of as a reification, or encapsulation, of a highly compressed process.[24] In the successive theorems in his Chapter 2, Cauchy seeks to collapse the initial articulation of his infinitesimal α as a temporally-deployed *process*, by deliberately leaving out the implied index (i.e. label of the terms of the sequence), and by explicitly specifying and emphasizing a rival index: the exponent in a power α^n of the infinitesimal, thought of as an infinitesimal of a higher and higher order.

We will examine Cauchy's approach further in Sect. 2.3 by means of a detailed analysis of his text.

2.3 An Analysis of *Cours d'Analyse* and Its Infinitesimals

We will refer to the pages in the *Cours d'Analyse* Cauchy (1821) itself, rather than the collected works.

Cauchy's Chapter 2, section 1 starts on page 26. Here Cauchy writes that a variable quantity becomes *infinitely small* if, etc. Here "infinitely small" is an adjective, and is not used as a noun-adjective pair.

On page 27, Cauchy employs the noun-adjective combination, by referring to "infinitely small quantities" (*quantités infiniment petites*, still in the feminine). He denotes such a quantity α. Note that the index in the implied sequence is suppressed (namely, α appears without a lower index).[25]

On page 28, he introduces a competing numerical index, namely the exponent, by forming the infinitesimals

$$\alpha, \alpha^2, \alpha^3, \ldots$$

By the time we reach the bottom of the page (fourth line from the bottom), he is already employing "infinitely small" as a *noun* in its own right: *infiniment petits* (in the masculine plural).

On page 29, Theorem 1 asserts that a highest-order infinitesimal will be smaller than all the others (infinitesimals are consistently referred to in the masculine). The theorem has not yet chosen a *letter* label for the competing index (i.e. order of infinitesimal).

Still on page 29, Theorem 2 for the first time introduces a label for the order of the infinitesimal, namely the letter n, as in

$$\alpha^n.$$

On page 30, Theorem 3 introduces several different letter indices:

$$n, n', n'' \ldots$$

for the orders of his infinitesimals. The theorem concerns the order of the sum, again compelling the student to focus on the competing orders (at the expense of the suppressed index of the "variable quantity" itself).

Still on page 30, Theorem 4 introduces the terminology of *polynomials* in α, and describes their orders $n, n', n'' \ldots$ for the first time as a "sequence". We now have two "sequences": (the variable quantity) α itself, whose index is implicit (was never labeled), and the sequence of orders, which are both emphasized and elaborately labeled using "primes" $'$ and double primes $''$.

[24] In the education literature, such a compressing phenomenon is studied under the name of procept (process+concept) (Gray and Tall 1994), encapsulation (Dubinsky 1991), and reification (Sfard 1991).

[25] Note that Cauchy uses lower indices to indicate terms in a sequence in his proof of the intermediate value theorem (Cauchy 1821, Note III, p. 462).

Table 1 Cauchy's first two definitions of continuity in 1821 are of the form *"if Δx is ..., then Δy is ..."*. Note the prevalence of the term "infinitesimal"

	Independent variable increment (Δx)	Dependent variable increment (Δy)
Cauchy's first definition	Infinitesimal	Variable tending to zero
Cauchy's second definition	Infinitesimal	Infinitesimal

In all these theorems, it is the *asymptotic* behavior of null sequences that is constantly emphasized, which suggests that Cauchy might have found it perfectly natural to identify/collapse sequences that agree almost everywhere. Terminology *finit par être*, *finit par devenir* (suggestive of such collapse) is employed repeatedly.

On pages 31 and 32, three additional theorems and one corollary are stated, for a total of eight results on the asymptotic behavior of such null sequences.

By the time Cauchy reaches Section 2 of Chapter 2 on page 34 (concerning continuity of functions), he has already encapsulated the *process* implied in the notion of "variable quantity", into the concept/masculine noun *infiniment petit*. When he evokes an infinitely small x-increment α, only a stubborn Weierstrassian will refuse to interpret his α as a concept/noun. The *second* definition (out of the three definitions of continuity given here) is the one Cauchy italicizes, implying it is the main one. Here both the x-increment and the y-increment are described as infinitely small increments (see Table 1).

In this context, the verb "become" is being used in two different senses:

(a) the terms in the sequence *become* smaller than any number;
(b) the encapsulating sense of a process being compressed into (and thus *becoming*) a concept/noun,

as analyzed by Sad et al. (2001), who employed the terminology of a *transformation of essence*. It is interesting to note that Bolzano fought against the sense (a), by suppressing the parametrisation altogether, and viewing the null sequence as a *set* (perhaps he was influenced by Zeno paradoxes), but not against the sense (b).

Note Cauchy's emphasis on the *noun* aspect of his infinitesimals:

> Lorsque les valeurs numériques[26] successives d'une même variable décroissent indéfiniment, de manière à s'abaisser au-dessous de tout nombre donné, cette variable devient ce qu'on nomme *un infiniment petit* ou une quantité infiniment petite. Une variable de cette espèce a zéro pour limite (Cauchy 1821, p. 4).

The use of the noun, *un infiniment petit*, makes it difficult to interpret the "becoming" in the sense (a) above; rather, the definition requires sense (b) to be grammatically coherent. Here the variable [quantity] *becomes* a masculine noun: *un infiniment petit*. Cauchy is very precise here: it is the *limit* of the variable that's zero. The variable itself *becomes* an *infiniment petit*. Cauchy wrote neither that a variable *is* an infinitesimal, nor that the limit of the infinitesimal is zero, but rather that the limit of the *variable* is zero, cf. Sad et al. (2001, pp. 301–302).

Once the use of *infiniment petit* as a noun is established, Cauchy freely uses the term interchangeably as a noun or as an adjective.

Cauchy's *Cours d'Analyse* presented only a theory of infinitesimals of polynomial rate of growth as compared to a given α. His theory of infinitesimals of more general order, and its

[26] The meaning of the expression *valeur numérique* is subject to debate; see next section and footnote 42.

influence on later authors such as E. Borel, was already discussed in Sect. 1;[27] see Sect. 4 for a broader historical perspective. In Sect. 2.4, we will discuss a specific application Cauchy makes of his infinitesimals, in the context of the sum theorem.

2.4 Microcontinuity and Cauchy's Sum Theorem

How is Cauchy able to define concepts such as uniform continuity and uniform convergence in terms of a *single* variable, unlike standard definitions thereof which call for a *pair* of variables?

Let x be in the domain of a function f, and consider the following condition, which we will call *microcontinuity* at x:

> "if x' is in the domain of f and x' is infinitely close to x, then $f(x')$ is infinitely close to $f(x)$".

Then ordinary continuity of f is equivalent to f being microcontinuous on the Archimedean continuum (A-continuum for short), i.e., at every point x of its domain in the A-continuum. Meanwhile, uniform continuity of f is equivalent to f being microcontinuous on the Bernoulian continuum (B-continuum for short), i.e., at every point x of its domain in the B-continuum (the relation of the two continua is discussed in more detail in Appendix A).

Consider, for example, the function $\sin(1/x)$ defined for positive x. The function fails to be uniformly continuous because microcontinuity fails at a positive infinitesimal x (due to ever more rapid oscillation). The function x^2 fails to be uniformly continuous because of the failure of microcontinuity at an infinite member of the B-continuum.

A similar distinction exists between pointwise convergence and uniform convergence.[28] Which condition did Cauchy have in mind in 1821? Abel interpreted it as convergence on the A-continuum, and presented "exceptions" (what we would call today counterexamples) in 1826. Additional such exceptions were published by Seidel and Stokes in the 1840s.

Cauchy's contemporary E. G. Björling anticipated the B-continuum definition in a remarkable passage dating from 1852:

> when one comes to show that in a series, of which the terms are functions of a quantity x, converges [...] for each given value of x up to a certain limit X, it is not necessary to believe that the series continues necessarily to converge [...] for values of x indefinitely close to that limit (Björling 1852, p. 455, as cited by Grattan-Guinness 1987, p. 232).

Here the "limit" X is clearly a member of the usual Archimedean continuum; but what is the nature of the x mentioned at the end of Björling's phrase? Bjorling's reference to convergence at x implies that x is understood to be an individual/atomic element of the domain; yet speaking of x as being "indefinitely close" to X is meaningless in the context of an A-continuum. One faces a stark choice of either toeing the triumvirate line on the "true" continuum and therefore finding Björling's arguments "absurd", as Pringsheim did Pringsheim (1897, p. 345);[29] or, removing the blinders so as to envision a mid-19th century anticipation of an infinitesimal-enriched continuum.

Cauchy clarified/modified his position on the sum theorem in 1853.[30] In his text Cauchy (1853), he specified a stronger condition of convergence on the B-continuum, including

[27] See also the main text following footnote 8.

[28] See e.g., Goldblatt (1998, Theorem 7.12.2, p. 87).

[29] As cited by Grattan-Guinness (1987, p. 233).

[30] Grattan-Guinness (1987) and Bråting (2007) argue that Cauchy was influenced by Björling.

at $x = \frac{1}{n}$ (explicitly mentioned by Cauchy). The stronger condition bars Abel's counterexample.

To give a more detailed explanation of Cauchy's (1987) text, note that Cauchy's approach is based on two assumptions which can be stated in modern terminology as follows:

(1) when you have a closed expression for a function, then its values at "variable quantities" (such as $x = \frac{1}{n}$) are calculated by using the same closed expression as at real values;
(2) to evaluate a function at a variable quantity generated by a sequence, one evaluates term-by-term.

Now in 1853 Cauchy analyzed Abel's counterexample (without mentioning Abel's name) by first writing down the closed form of the sum of the *remainder* of the series. This is given by a certain integral. He proceeded to evaluate it at an infinitesimal $x = \frac{1}{n}$ using assumption (1). Concretely, he substituted the values $\frac{1}{n}$ into the closed expression (integral) using assumption (2). The sequence he got was *not* a null sequence. He concluded that the remainder term at this particular infinitesimal is not an infinitesimal. Hence, he concluded, the "always" part of his hypothesis is not satisfied by Abel's example.

Thus, Cauchy's clarification/modification from 1853 amounts to requiring convergence on an incipient form of a B-continuum. Some historians acknowledge his clarification/modification, and interpret it as the addition of the condition of uniform convergence, while adhering to an A-continuum framework. The recent text Katz and Katz (2011b) argued that such an interpretation is problematic. Namely, Cauchy states the condition in terms of a *single* variable, whereas the traditional definition of uniform continuity or convergence in the context of an A-continuum necessarily requires *a pair* of variables. Cauchy specifically evokes $x = \frac{1}{n}$ where Abel's "exception/counterexample" fails to converge. The matter is discussed in detail by Bråting (2007).

2.5 Five Common Misconceptions in the Cauchy Literature

Both Laugwitz and Hourya Benis Sinaceur (Sinaceur 1973) have exposed a number of misconceptions in the literature concerning Cauchy's foundational work. Five of the most common ones are reproduced below in italics.

1. *Bolzano, Cauchy, and Weierstrass were all gardeners who contributed to the ripening of the fruit of the notion of limit.*

Here the implicit assumption is that the Weierstrassian epsilontic notion of "limit" in the context of an Archimedean continuum is the centerpiece of any possible edifice of analysis. Such an assumption is questionable on two counts. First, as Felix Klein pointed out in 1908, there are two parallel threads in the development of analysis, one based on an Archimedean continuum, and the other exploiting an infinitesimal-enriched continuum.[31] One risks pre-judging the outcome of any analysis of Cauchy by postulating that he is working in the Archimedean thread. The second implicit assumption is that the Weierstrassian notion of limit is central in Cauchy. Thus, Boyer (1949, p. 277) postulates that Cauchy is working with a notion of limit similar to the Weierstrassian one. This requires further argument, and at least at first glance is incorrect: Cauchy emphasizes infinitesimals as a foundational notion, but he never emphasizes limits as a foundational notion. Thus, in Cauchy's definition of continuity the word "limit" does occur, but only in the sense of the "endpoint" of the interval of definition of the function, rather than the behavior of its values.[32]

[31] See material in the main text around footnote 14.

[32] The misconception may be found, for instance, in J. Gray, see main text around footnote 11.

2. *Cauchy, along with other mathematicians, abandoned infinitesimals in favor of other more rigorous notions.*

During the period 1814–1820, Cauchy appears to have been ambivalent about infinitesimals. Starting in about 1821, he uses them with increasing frequency both in his textbooks and his research publications, and insists on the centrality of infinitesimals as a foundational notion; see Sect. 4 for a related chronology.

3. *Cauchy was forced to teach infinitesimals at the Ecole.*

The intended implication appears to be that Cauchy only used infinitesimals because of the pressure from the Ecole administration. During the period 1814–1820 there were some tensions with the administration over the delayed appearance of infinitesimals in the syllabus, see Gilain (1989). At any rate, Cauchy continued using infinitesimals throughout his career and long after completing his teaching stint at the Ecole in 1830. Thus he reproduces his 1821 definition of continuity (in terms of infinitesimals) as late as 1853, in his text on the sum theorem (Cauchy 1853), see Sect. 2.4.

4. *Cauchy based his infinitesimals on the notion of limit.*

This is an ambiguous claim, and essentially a play on words on the term "limit". The modern audience understands "limit" as a Weierstrassian epsilontic notion. If this is what is claimed, then the claim is false. As far as the kinetic notion of limit that Cauchy does mention in discussing a variable quantity approaching a limit, it is conspicuously absent in Cauchy's discussion of infinitesimals. Thus, rather than infinitesimals being based on the notion of limit, it is the notion of a variable quantity that's primitive, and both infinitesimals and limits are defined in terms of it (see Sect. 3.1 for a more detailed discussion). Note again that the term "limit" does appear in Cauchy's definition of continuity, but in an entirely different sense, namely endpoint of the interval where the function is defined.[33]

5. *Cauchy introduced rigor into calculus that anticipates the rigor of Weierstrass.*

While Cauchy certainly emphasizes rigor, postulating a continuity between Cauchy's rigor and Weierstrassian rigor is a methodological error.[34] To Cauchy, rigor meant abandoning the principle of the "generality of algebra" as practiced by Euler, Lagrange, and others, and its replacement by geometry–and by infinitesimals.

3 Comments by Modern Authors

In this section, we analyze some comments by modern authors related to Cauchy's foundational contribution.

[33] To elaborate, we will note that the purported opposition between infinitesimals and limits (a frequently found claim in the literature) is a conceptual error in its own right. The opposition is not between limits and infinitesimals. Limits are present in both approaches. In the infinitesimal approach, limits can be defined in terms of the standard part function, and the latter in terms of limits. The true opposition is between infinitesimals, on the one hand, and epsilon-delta with its quantifier complications, on the other. The preoccupation with the word "limit" can be exposed as a purely linguistic one by the following thought experiment, based on Pourciau's lucid analysis, see footnote 15. Imagine that Newton had developed an "ult" notation for his ultimate ratios, e.g., $\mathrm{ult}_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for the derivative, and $\mathrm{ult}_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$ for the integral. Post-Weierstrassian historians would then perhaps be more moderate in their enthusiasm for the vague musings about limits (incorrectly) attributed to d'Alembert, see Sect. 3.3.

[34] See Klein's comments discussed in the main text around footnote 14, concerning the two parallel strands for the development of analysis.

3.1 Gilain's Limit

C. Gilain affirms the following:

> On sait que Cauchy définissait le concept d'infiniment petit à l'aide du concept de limite, qui avait le premier rôle[35] (voir Analyse algébrique, p. 19…)" (Gilain 1989, footnote 67).

Here Gilain is referring to Cauchy's Collected Works, Série 2, Tome 3, p. 19, corresponding to (Cauchy 1821, p. 4). Both of Gilain's claims are erroneous, as we now show. Cauchy starts by discussing *variable quantities* as a primary notion, in the following terms:

> On nomme quantité *variable* celle que l'on considère comme devant reçevoir successivement plusieurs valeurs différentes les unes des autres.

Next, Cauchy exploits his primary notion to evoke his kinetic concept of limit[36] in the following terms:

> Lorsque les valeurs successivement attribuées à une même variable s'approchent indéfiniment d'une valeur fixe, de manière à finir par en différer par aussi peu qu'l'on voudra, cette dernière est appelée la *limite* de toutes les autres.

Finally, Cauchy proceeds to define infinitesimals in the following terms:

> Lorsque les valeurs numériques successives d'une même variable décroissent indéfiniment, de manière à s'abaisser au-dessous de tout nombre donné, cette variable devient ce qu'on nomme *un infiniment petit* ou une quantité infiniment petite. Une variable de cette espèce a zéro pour limite.[37]

Thus, Cauchy defined both infinitesimals and limits in terms of variable quantities. Neither is the limit concept primary, nor are infinitesimals defined in terms of limits, contrary to Gilain's claims.[38]

3.2 Felscher's *Bestiarium infinitesimale*

We have argued for an interpretation of Cauchy's foundational stance that endeavors to take Cauchy's infinitesimals at their face value. Such an interpretation has not been without its detractors. A decade ago, Felscher (2000) set out to investigate Cauchy's continuity, in an 18-page text, marred by an odd focus on d'Alembert.[39] To be sure, it is both legitimate and necessary to examine Cauchy's predecessors, including d'Alembert, if one wishes to understand Cauchy himself. Indeed, a debate of long standing (over a century long, in fact)

[35] "We know that Cauchy defined the concept of infinitely small by means of the concept of limit, which played the primary role."

[36] Such a concept is similar to Newton's, see footnote 15.

[37] This definition was already analyzed in Sect. 2.3, see main text around footnote 26.

[38] For a similar misconception in Smithies, see footnote 4.

[39] Klein (1932, p. 103) discusses the error in d'Alembert's proof of the fundamental theorem of algebra, first noticed by Gauss.

had opposed two rival methodologies in the study of the foundations of the new science of Newton and Leibniz:

(A) a methodology eschewing infinitesimals; and
(B) a methodology favoring them.[40]

It is legitimate to ask which of the two methodologies is the one that underpins Cauchy's oeuvre. However, Felscher's conceptual framework is flawed in a fundamental way. The outcome of his investigation is predetermined from the outset by the following two factors:

(1) Felscher's exclusive focus on d'Alembert,[41] one of the radical adherents of the A-methodology, and
(2) Felscher's *postulating* a methodological continuity between the work of d'Alembert and Cauchy (see also Sect. 3.3).

Displaying a masterly command of scholarly Latin, Felscher offers the reader a glimpse of the *bestiarium infinitesimale* in section 6 of his essay, starting on page 856. The punchline comes in the middle of page 857, where Felscher points out that Cauchy refers specifically to *numerical* values of his variables, the latter being described by Cauchy as *becoming* infinitesimals.

The adjective *numerical* is linked etymologically to the noun *number*. Cauchy's numbers (unlike his *quantities*) are certainly appreciable (i.e., neither infinitesimal nor infinite, nor even negative), as can be seen by reading the first page *Préliminaires* of his 1821 *Cours D'Analyse*. Felscher concludes that the variables assume only appreciable values, but not non-Archimedean ones.[42]

Felscher's etymological insight offers a refutation of the Luzin hypothesis,[43] to the effect that Cauchy variables may pass through non-Archimedean values on their way to zero. Has Felscher shown that Cauchy's *bestiarium infinitesimale* is in fact uninhabited?

Hardly so. While chasing out the infinitesimal mouse of Luzin's hypothesis[44] for the Cauchy variable, Felscher missed the elephant of the possibility of the variable *becoming* an infinitesimal through a process of reification (see Sect. 2.2).

On page 846, Felscher quotes an agitated passage from d'Alembert's 1754 article. D'Alembert attacks the *obscurity*, and even the *falsehood* of a definition of infinitesimals attributed to unnamed geometers, and sums up his thesis by accusing such geometers of *charlatanerie*, a term ably translated as *quackery* by Felscher, who sums up as follows:

[40] See Felix Klein's comments discussed in the main text around footnote 14.

[41] Felscher mentions Euler and the Bernoullis in his section entitled *D'Alembert's program*, but says not a word about them in his article.

[42] Note that Schubring (2005, p. 446), in footnote 14, explains Cauchy's term *numerical value* as what we would call today the *absolute value*. Fisher (1979, p. 262) interprets Cauchy's definition accordingly, so as to allow room for infinitesimal values of Cauchy's variables. See also footnote 44 on Cleave.

[43] Luzin himself, in fact, similarly rejected non-Archimedean time, as discussed by Medvedev (1993).

[44] Luzin was probably not the first and surely not the last to formulate a hypothesis to the effect that Cauchy's variable quantities pass through infinitesimal values on their way to zero. Lakatos (1978, p. 153) speculates that Cauchy's variables "ran through Weierstrassian real numbers *and* infinitesimals", while J. Cleave (who edited Lakatos's essay for publication in the *Mathematical Intelligencer*) in footnote 18* in Lakatos (1978, p. 159) disagrees, limiting Cauchy variables to sequences of Weierstrassian reals (Cleave quotes the relevant passage on *numerical values* but does not analyze it here). Cleave alludes to the etymological point in Cleave (1979, p. 268), where he disagrees with Fisher on this point.

> Reading these words today we may get the impression that they were written at the time of Weierstrass or Cantor,[45] or even by a contemporary mathematician.[46]

It is sobering to realize that, forty years after A. Robinson, a logician named Walter Felscher still conceived of the history of analysis in terms of a triumphant march out of the dark age of the infinitesimal, and toward the yawning heights of Weierstrassian epsilontics.

D'Alembert's verbal excesses merely put in relief the fact that no such rhetoric is to be found anywhere in either Cauchy or Bolzano. Felscher presents a convincing case that d'Alembert was opposed to infinitesimals. Felscher's title *Bolzano, Cauchy, epsilon, delta*[47] could therefore have pertinently been replaced by *D'Alembert, Weierstrass, epsilon, delta*, as the case for Bolzano's opposition to infinitesimals can similarly be challenged.[48] Indeed, as Lakatos points out,

> [Bolzano] was possibly the only one to see the problems related to the difference between the two continuums: the rich Leibnizian continuuum and, as he called it, its 'measurable' subset–the set of Weierstrassian real numbers. Bolzano makes it very clear that the field of 'measurable numbers'[49] constitutes only an Archimedean subset of a continuum enriched by non-measurable—infinitely small or infinitely large—quantities (Lakatos 1978, p. 154).

Kurepa (1982, p. 664) provides some details on Bolzano's use of infinitesimals. As far as epsilon-delta techniques are concerned, a case for Bolzano's anticipation thereof is far more convincing than for Cauchy. [50]

Felscher's intriguing parenthetical remark indicates that he was more sensitive to Cauchy's language than numerous Cauchy historians:

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45 Cantor was indeed a worthy heir to d'Alembert's anti-infinitesimal vitriol. Cantor dubbed infinitesimals the *cholera bacillus* of mathematics, see Dauben (1995, p. 353, 1996, p. 124), Meschkowski (1965, p. 505). This was perhaps the most vitriolic opposition to the B-continuum (see Appendix A) prior to Errett Bishop's *debasement of meaning*, a term he applied to classical mathematics in general in 1973 (Bishop 1985), and to infinitesimal calculus à la Robinson in particular, in 1975 (Bishop 1975), see Katz and Katz (2011a,d) for details.

46 It is worth pondering which contemporary mathematician (known for anti-infinitesimal vitriol) Felscher may have had in mind here, given his interest in intuitionistic logics Felscher (1985, 1986), see also Katz and Katz (2011a).

47 Apparently, a kind of a mantra: *Bolzano, Cauchy, epsilon, delta; Bolzano, Cauchy, epsilon, delta; ...* which, repeated sufficiently many times, would lead one to accept Felscher's reduction of Cauchy's continuum to an A-continuum (see Appendix A).

48 In discussing Bolzano's attitude toward infinitesimals, we have to distinguish between the early Bolzano and the late Bolzano. The early Bolzano defines the "Infinitely small" as "variable quantities" in the following terms: A quantity is infinitely small if it becomes less than any given quantity (here Bolzano does not speak of "values" [but naturally he thought of them]). The late Bolzano definies infinitely small (and infinitely large) *numbers*; one of them is $1/(1+1+1+\ldots)$ (infinitely many terms). We are grateful to D. Spalt for this historical clarification.

49 *Measurable number* is Bolzano's term for appreciable number (no relation to Lebesgue-measurability). Here Bolzano foreshadows Björling's dichotomy (see Bråting 2007), which can be analyzed in terms of A- and B-continua (see Appendix A).

50 Hourya Benis Sinaceur wrote: "les oeuvres de Bolzano et de Cauchy représentent deux courants distincts, hétérogenès à l'origine, et qui ne se rencontrent pas avant les années 1870, avec les travaux de Weierstrass, Cantor, etc. Seule l'illusion rétrospective et une connaissance indirecte des textes permettent de les amalgamer dans un même cours ininterrompu supposé traverser tout le XIX^e siècle" (Sinaceur 1973, p. 111), and cites a letter of H. A. Schwarz to Cantor [105] to the effect that "la méthode de démonstration [...] de Weierstrass est un développement des principes de Bolzano" (Sinaceur 1973, p. 112).

> it is left open whether a *quantité variable*, with an assignment converging to zero, actually *is* or only *becomes* a *quantité infiniment petite* (Felscher 2000, p. 850).

On page 851, Felscher presents an analysis of Cauchy's use of an infinitesimal quantity, denoted i, in differentiating an exponential function. Here Felscher's additional parenthetical remark, to the effect that "notational confusion arises from denoting both the variable i and its values by the same letter", is an unjustified criticism of Cauchy. The criticism underscores Felscher's insensitivity toward the dynamic aspect of Cauchy's infinitesimal i, when individual values are irrelevant in the context of the dynamism of the encapsulation taking place whenever Cauchy evokes an infinitesimal. Otherwise Felscher's analysis is unexceptionable, save for a *non-sequitur* of a conclusion:

> No 'infinitesimal' non-Archimedean numbers are ever used by Cauchy for his *quantités infiniment petites*.

In reality, Cauchy's discussion of the derivative of the exponential function admits a number of possible interpretations.

On page 852, Felscher analyzes Cauchy's infinitesimals in modern terms:

> Using today's terminology, one would describe Cauchy's forms to be filled by assignments as functions, but in order to distinguish them from the actual functions subsequently considered by Cauchy, one might call them *functional germs*. [Emphasis in the original—authors]

Felscher mentions Cauchy's use of functional germs again on page 855. In his zeal to rename Cauchy's infinitesimals by employing a modern notion, so as bashfully to avoid the distasteful *infi* term, Felscher comes close to endorsing Laugwitz's "Cauchy numbers", similarly defined in terms of germs.

Thus, Laugwitz wrote:

> Every real function $f(u)$ defined on an interval $0 < u < p$ represents a Cauchy number. Two such functions $f(u)$ and $g(u)$ represent the same Cauchy number if and only if there is an interval $0 < u < q$ in which $f(u) = g(u)$ (Laugwitz 1997, p. 659).

Note that Laugwitz essentially defines his "Cauchy numbers" by exploiting the concept of the germ of a function; Laugwitz explicitly mentions *function germs* in Laugwitz (1987, p. 272). Felscher (2000, p. 858) leaves very little doubt as to how he felt with regard to Laugwitz infinitesimals:

> In this connection one must also mention certain articles and books by D. Laugwitz, in which [...] he develops his own 'mathematics of the infinitesimal' and uses it to interpret skillfully various aspects of the mathematics of the period from Euler to Cauchy. And so we have glanced at the *bestiarium infinitesimale*. [51]

On page 853, Felscher *omits* a crucial first phrase used by Cauchy in formulating his first definition of continuity. Namely, he omits Cauchy's phrase *stating that α is an infinitesimal*, corresponding to the upper-left entry in Table 1. [52]

On pages 854–855, Felscher makes the following statement concerning Cauchy's notion of *continuity*:

[51] The reviewer for MathSciNet concurred, see Sect. 3.4.

[52] This crucial detail leads Felscher to a further error of a conceptual nature, discussed below.

> Both Bolzano and Cauchy gave definitions of continuity which express today's […] continuity. Both made their definitions precise and used them in today's sense; both employed them by comparing numbers and their distances with the help of inequalities in order to prove important theorems in analysis. However, Cauchy defined and used the notion of limit, whereas Bolzano did not.

Felscher's assertion concerning continuity and inequalities is misleading. Generally speaking, Cauchy's limit concept is a kinetic one (see Sinaceur 1973), and is a derived notion, depending for its definition on the primary notion of a *variable quantity*. On occasion, Cauchy worked with real inequalities in proofs (see, e.g., Grabiner 1983), but he never gave such a definition of *continuity*.

On page 855, line 9, Felscher alleges that Cauchy's first definition of continuity is similar to Bolzano's, with the implication that the terminology of "infinitesimal" is not employed by Cauchy. Now the form in which Cauchy's definition was quoted by *Felscher* two pages earlier (see our comment above concerning Felscher's page 853) did not employ infinitesimals. But the form in which it appears in *Cauchy* did employ infinitesimals. [53]

The most remarkable aspect of Felscher's, unfortunately seriously flawed, essay is how close he comes to sensing the cognitive view of compression/encapsulation outlined in Subsect. 2.2:

> We speak of a function or a variable approaching some value *indéfiniment* (indefinitely); we imagine a limiting process. […] Thus far, ϵ and δ (and in case of sequences also n and N) appear as handles affixed to the stages of those infinite processes. It seems that if appropriately handled in our mental exercises, they enable us to use finitely many arguments to prove statements that, in the end, speak about all the stages of the infinite process (Felscher 2000, p. 858).

3.3 D'Alembert or de La Chapelle?

Felscher's text analyzed in Sect. 3.2 describes d'Alembert as "one of the mathematicians representing the heroic age of calculus" (Felscher 2000, p. 845). Felscher buttresses his claim by a lengthy quotation concerning the definition of the limit concept, from the article *Limite* from the *Encyclopédie ou Dictionnaire Raisonné des Sciences, des Arts et des Métiers* (volume 9 from 1765):

> On dit qu'une grandeur est la limite d'une autre grandeur, quand la seconde peut approcher de la première plus près que d'une grandeur donnée, si petite qu'on la puisse supposer, sans pourtant que la grandeur qui approche, puisse jamais surpasser la grandeur dont elle approche; ensorte que la différence d'une pareille quantité à sa limite est absolument inassignable (Encyclopédie, volume 9, page 542).

One recognizes here a kinetic definition of limit already exploited by I. Newton. [54] Whatever the merits of attributing visionary status to this quote, what Felscher overlooked is the fact that the article *Limite* was written by two authors. In reality, the above passage defining the concept of "limit" (as well as the two propositions on limits) did not originate with

[53] See Table 1 (for a summary of Cauchy's definitions) and footnote 52. Schubring (2005, p. 465) writes that J. Lützen's is the best analysis of continuity in Cauchy. Meanwhile, Lützen (2003, p. 166) states: "Cauchy […] gives two definitions, first one without infinitesimals, and then one using infinitesimals." The second claim is correct, but not the first.

[54] See footnote 15 on Pourciau's analysis.

d'Alembert, but rather with the encyclopedist Jean-Baptiste de La Chapelle. De la Chapelle was recruited by d'Alembert to write 270 articles for the *Encyclopédie*.

The section of the article contaning these items is signed (E) (at bottom of first column of page 542), known to be de La Chapelle's "signature" in the *Encyclopedie*. Felscher had already committed a similar error of attributing de la Chapelle's work to d'Alembert, in his work Felscher (1979). [55] Note that Robinson (1966, p. 267) similarly misattributes this passage to d'Alembert.

3.4 Brillouët-Belluot: Epsilon-Delta, Period

An instructive case study in a *bestiarium*-consignment attitude toward infinitesimals is the review of Felscher's text for Math Reviews, by one Nicole Brillouët-Belluot. The review contains not an inkling of the fact that the text in question is a broadside attack on scholars attempting to analyze Cauchy's infinitesimals seriously. The reviewer mentions the "epsilon-delta technique", and notes that Cauchy made his "definitions [of continuity] precise and used them in today's sense", but fails to mention that the definitions in question are *infinitesimal* ones, a fact not denied by Felscher (at least in the case of one of the definitions). Her review mentions d'Alembert, who does not appear in Felscher's summary, indicating that she had read the body of Felscher's text itself (rather than merely Felscher's summary).

Brillouët-Belluot notes that Felscher reports on how "limit was explained and used by d'Alembert and Cauchy". She reports neither on Felscher's extensive, and vitriolic, quotes from d'Alembert (including the dramatic phrase "the metaphysics and the infinitely small quantities, whether larger or smaller than one another, are totally useless in the differential calculus"), nor d'Alembert's colorful epithets like *charlatanerie*/quackery. The reviewer identified with Felscher's conclusions to such an extent that she chose to spare the Math Reviews reader the burden of infinitesimal quackery, judging that Felscher carried that burden once and for all, for the rest of us. [56]

3.5 Bos on Preliminary Explanation

Leibniz historian Bos acknowledged that Robinson's hyperreals provide a

> preliminary explanation of why the calculus could develop on the insecure foundation of the acceptance of infinitely small and infinitely large quantities (Bos 1974, p. 13).

3.6 Medvedev's Delicate Question

Reviewer Cooke (1980) notes that Medvedev (1987) "devotes considerable space to refuting the point of view of modern analysis as a thing waiting to be born, which earlier mathematicians were striving unsuccessfully to find and for lack of which they were merely groping in the dark. In particular he argues that the approaches of Cauchy and Weierstrass were so different from each other that it would be more accurate to speak of two systems of analysis."

F. Medvedev further points out that nonstandard analysis

> makes it possible to answer a delicate question bound up with earlier approaches to the history of classical analysis. If infinitely small and infinitely large magnitudes are [to

[55] We are grateful to D. Spalt for this historical clarification.

[56] Similar remarks apply to the review of Felscher's text by Siegmund-Schultze (2000) for Zentralblatt Math. Siegmund-Schultze was far less myopic in a later instance occasioned by a review of an English translation of the *Cours d'analyse*, see footnote 23.

> be] regarded as inconsistent notions, how could they [have] serve[d] as a basis for the construction of so [magnificent] an edifice of one of the most important mathematical disciplines? Medvedev (1987, 1998)

A powerful question, indeed. How do historians answer Medvedev's question?

3.7 Grabiner's "Deep Insight"

Not all scholars are satisfied with the *amazing-intuition-and-deep-insight* answer offered by J. Grabiner who writes:

> [M]athematicians like Euler and Laplace had a deep insight into the basic properties of the concepts of the calculus, and were able to choose fruitful methods and evade pitfalls (Grabiner 1983, p. 188)

How can deep insight manage to "evade pitfalls" if the foundations are regarded as inconsistent? Grabiner (1983, p. 189) further claims that,

> [s]ince an adequate response to Berkeley's objections would have involved recognizing that an equation involving limits is a shorthand expression for a sequence of inequalities—a subtle and difficult idea—no eighteenth century analyst gave a fully adequate answer to Berkeley.

This is an astonishing claim, which amounts to reading back into history, developments that came much later. Such a claim amounts to postulating the inevitability of a triumphant march, from Berkeley onward, toward the radiant future of Weierstrassian epsilontics. [57] The claim of such inevitability in our opinion is an assumption that requires further argument.

Berkeley was, after all, attacking the coherence of *infinitesimals*. [58] He was *not* attacking the coherence of some kind of incipient form of Weierstrassian epsilontics and its inequalities. Isn't there a simpler answer to Berkeley's query, in terms of a passage from a point of B-continuum (see Appendix A), to the infinitely close point of the A-continuum, namely passing from a variable quantity to its limiting constant quantity?

A related attitude on the part of Felscher is discussed in Sect. 2.3.

Like Felscher, Grabiner (1983, p. 190) suppresses Cauchy's reference to an infinitesimal increment of the independent variable when citing Cauchy's first definition (see Table 1 in Sect. 2.3 above), thereby managing to avoid discussing Cauchy's infinitesimals altogether. We encounter the oft-repeated claim about "the same confusion between uniform and pointwise convergence" [59] (Grabiner 1983, p. 191). Her discussion of Cauchy's rigor does not mention that what rigor meant to Cauchy was the replacement of the principle of the generality of algebra, by geometry, including infinitesimals. Grabiner correctly points out (Grabiner 1983, p. 193) that "Mathematicians are used to taking the rigorous foundations for calculus for granted." She concludes: "What I have tried to do as a historian is to reveal what went into making up that great achievement." What we have tried to do is to introduce a necessary correction to a post-Weierstrassian reading of Cauchy, influenced by an automated

[57] Hourya Benis Sinaceur similarly mocks "l'hypothèse d'un développement monolithique, non différencié, continue, univoque de l'analyse au XIX^e siecle. Le filon, perdu avec Bolzano, est heureusement retrouvé par Cauchy, qui le redécouvre, l'élargit, l'exploite plus amplement, le transmet à Weierstrass, etc." (Sinaceur 1973, p. 103), and names (Boyer 1959, p. 271) as one of the culprits in perpetuating such a myth.

[58] Berkeley's criticism is dissected into its metaphysical and logical components by Sherry (1987).

[59] See Subsect. 2.4 for a detailed discussion of the controversy over the "sum theorem".

infinitesimal-to-limits translation originating no later than Boyer (1949, p. 277), see Katz and Katz (2011b) for additional details.

3.8 Devlin: Will the Real Cauchy–Weierstrass Please Stand Up?

The following exchange between the second-named author and Keith Devlin, co-founder and Executive Director of Stanford University's H-STAR institute, took place in may 2011. The exchange was occasioned by Devlin's article "Will the real continuous function please stand up?" [60] The article refers to an alleged "Cauchy–Weierstrass definition of continuity".

MK: I read with interest your online article on "continuous function please stand up". I am not sure which Cauchy–Weierstrass definition you are referring to. What exactly *was* Cauchy's definition of continuity? I think I know what Weierstrass's was.

KD: See [61] http://www.maa.org/pubs/Calc_articles/ma002.pdf

MK: But Grabiner does not say that Cauchy gave an epsilontic definition of continuity. I wonder if there is a reason for that. Do you know what Cauchy's definition of continuity was?

KD: Actually, she does say that: "Delta-epsilon proofs are first found in the works of Augustin-Louis Cauchy (1789–1867). This is not always recognized, since Cauchy gave a purely verbal definition of limit, which at first glance does not resemble modern definitions." In the UK when I was a student, the standard term to refer to the usual ϵ-δ definition was "the Cauchy–Weierstrass definition", and by my read of Grabiner that description seems appropriate.

MK: But my question concerned the definition of *continuity*. In the UK, did they refer to a Cauchy–Weierstrass definition of continuity?

KD: It's the same. $f(x)$ is continuous at a if and only if $f(a)$ is defined and $\lim_{x\to a} f(x) = f(a)$.

MK: Keith, this is *your* definition of continuity. It is *not* Cauchy's definition of continuity, for several reasons:

(1) Cauchy did not at any time work with "continuity at a point", it is always continuity "between two limits", i.e. on an interval.

(2) Cauchy's definition is invariably an infinitesimal one: "an infinitesimal x-increment always produces an infinitesimal y-increment".

(3) Elsewhere, Cauchy is working for the most part with a kinetic definition of limit, akin to that found in Newton. [62] Thus, the primary notion is that of a variable quantity. Both infinitesimals and limits are defined in terms of variable quantity.

Could the notion of a "Cauchy–Weierstrass definition of continuity" be an ahistorical blunder, regardless of whether it was adopted by the mathematicians at a UK college?

KD: That was actually the way continuity was taught in the UK. To my mind, once you have the basic epsilon-delta idea, which is a clever static capture of the dynamic concept, everything just drops out. It comes down to that one idea. And according to that Grabiner article, that was due to Cauchy and then Weierstrass. Of course, once we had that definition, our entire conception changed, and it may be that it is only from a modern

[60] The article is online at http://www.maa.org/devlin/devlin_11_06.html.

[61] The link provided is a link to an electronic version of Grabiner's article Grabiner (1983).

[62] See footnote 15.

perspective that we can clearly see that there is really just one fundamental notion there. That happens all the time, of course. Our present day conception of Newton's differential calculus is not the one Newton had. [63] Sounds like you are trying to get at how the people at the time conceived things.

MK: What you seem to be saying is that, even though Cauchy himself said nothing of the sort regarding continuity, from a certain "modern perspective" is it proper to interpret Cauchy in a certain way, now that in retrospect we know that the epsilontic idea is the "fundamental notion". This is precisely the ideology that I seek to refute. Felix Klein pointed out in 1908 (in the original edition of his "Elementary mathematics from an advanced viewpoint") that there are really *two* parallel strands to the development of analysis, the epsilontic one and the infinitesimal one. Emile Borel in 1902 developed further du Bois-Reymond's theory of infinitesimal-enriched continua in terms of growth rates of functions, and specifically refers to Cauchy's work in this direction from 1829 as inspiration. Note that Klein and Borel said this well before Robinson. Viewed as the "father" of the infinitesimal approach ultimately vindicated by Robinson, Cauchy needn't suffer a forced explanation as a proto-Weierstrassian. Rather, his approach to infinitesimals was further developed by Stolz, du Bois-Reymond, Borel, Levi-Civita, and others around the turn of the century. When Skolem developed first non-standard models of arithmetic in 1934, he was inspired partly by du Bois-Reymond's work, according to Robinson. In 1948, Hewitt first developed hyperreal fields using ultraproducts. In 1955, Łoś proved his theorem for ultraproducts, which implies the transfer principle, which is the mathematical implementation of Leibniz's heuristic "law of continuity": whatever succeeds for the finite, should also succeed for the infinite. Isn't this lineage of infinitesimal calculus more plausible than an alleged "Cauchy–Weierstrass definition of continuity"?

KD: Ah, I assume you are a historian of mathematics. For sure my perspective is very much that of mathematicians, who if they have any interest in history (and I do) it is precisely from the perspective you describe, tracing back the origins of the now accepted concepts in terms that we now use. For instance, we interpret Newton's work on calculus as applied to continuous functions of a real variable, when that is not at all what Newton was doing. (Actually, I don't think we can really understand what he was doing, since we are rooted in a modern perspective.) Good luck with your investigations. [64]

To put the above exchange in perspective, it may be useful to recall Grattan-Guinness's articulation of a historical reconstruction project in the name of Freudenthal (1971), in the following terms:

> it is mere feedback-style ahistory to read Cauchy (and contemporaries such as Bernard Bolzano) as if they had read Weierstrass already. On the contrary, their own pre-Weierstrassian muddles [65] need historical reconstruction (Grattan-Guinness 2004, p. 176).

3.9 Grattan-Guinness on Beautiful Mathematics

On the subject of Robinson's theory, Grattan-Guinness comments as follows:

[63] Pourciau refutes this view; see footnote 15.

[64] Devlin's article acknowledges that it is based on Núñez et al. (1999), who similarly employ the term "Cauchy–Weierstrass definition of continuity".

[65] Grattan-Guinness's term "muddle" refers to an irreducible ambiguity of historical mathematics such as Cauchy's sum theorem of 1821.

> I made no mention of non-standard analysis in my book, for it was obvious to me that this very beautiful piece of mathematics had nothing to tell us historically (Grattan-Guinness 1978/1979, p. 247).

When Grattan-Guinness announced that Robinson's construction of a non-Archimedean extension of the reals "bears no resemblance to past arguments in favor of infinitesimals" (Grattan-Guinness 1978/1979, p. 247), he was only telling part of the story. True, the construction favored by Robinson exploited powerful compactness theorems [66] originating with Malcev (1936) and eschewed the sequential approach. On the other hand, the ultrapower construction of the hyperreals, pioneered by Hewitt (1948) in 1948 and popularized by Luxemburg (1964) in 1962, is firmly rooted in the sequential approach, and hence connects well with the kinetic vision of Cauchy, shared by L. Carnot. [67]

4 Timeline of Modern Infinitesimals from Cauchy Onward

The historical sequence of events, as far as continuity is concerned, was as follows: [68]

(1) first came Cauchy's infinitesimal definition, namely "infinitesimal x-increment always produces an infinitesimal y-increment";
(2) then came the Dirichlet/Weierstrass-style nominalistic reconstruction of the original definition in terms of real inequalities, dispensing with infinitesimals;
(3) Cauchy's work influenced investigations in infinitesimal-enriched continua by du Bois-Reymond, E. Borel, and others at the turn of the century, see Table 2 (a more detailed account appears below).

The historical *priority* of the infinitesimal definition is clear; what is open to debate is the role of the modern definition in *interpreting* the historical definition. The common element here is the *null sequence*, a basis both for Cauchy infinitesimals, and, via intermediate developments in growth rates of functions as explained below, for ultrapower-based infinitesimals. [69]

4.1 From Cauchy to du Bois-Reymond

Cauchy's theory of arbitrary orders of magnitude for his infinitesimals was a harbinger, not of Weierstrassian epsilontics, but of later theories of infinitesimal-enriched continua as developed by Stolz (1885); du Bois-Reymond (1882) and others. Emile Borel's appreciation of the essential continuity between Cauchy's theory and that of his late 19th century heirs was already discussed in Subsect. 2.3. [70] Robinson (1966, pp. 277–278) traces the evolution of the infinitesimal ideal from Cauchy to du Bois-Reymond and Stolz, to Skolem. [71]

[66] Robinson uses instead the term "finiteness principle of lower predicate calculus" (Robinson 1966, p. 48) for what is known today as the compactness theorem.

[67] Some details on the ultrapower construction appear in Appendix A.

[68] We leave out Bolzano's contribution which, while prior to Cauchy's, did not exert any influence until the 1860s.

[69] Modulo suitable foundational material, one can ensure that every hyperreal infinitesimal is represented by a null sequence; an appropriate ultrafilter (called a P-point) will exist if one assumes the continuum hypothesis, or even the weaker Martin's axiom (see Cutland et al. (1988) for details).

[70] See the main text following footnote 8.

[71] See the main text around footnote 9.

Table 2 Timeline of modern infinitesimals from Cauchy to Nelson

Years	Author	Contribution
1821	Cauchy	Infinitesimal definition of continuity
1827	Cauchy	Infinitesimal delta function
1829	Cauchy	Defined "order of infinitesimal" in terms of rates of growth of functions
1852	Björling	Dealt with convergence at points "indefinitely close" to the limit
1853	Cauchy	Clarified hypothesis of "sum theorem" by requiring convergence at infinitesimal points
1870-1900	Stolz, du Bois-Reymond, and others	Infinitesimal-enriched number systems defined in terms of rates of growth of functions
1902	Emile Borel	Elaboration of du Bois-Reymond's system
1910	G. H. Hardy	Provided a firm foundation for du Bois-Reymond's orders of infinity
1926	Artin–Schreier	Theory of real closed fields
1930	Tarski	Existence of ultrafilters
1934	Skolem	Nonstandard model of arithmetic
1948	Edwin Hewitt	Ultrapower construction of hyperreals
1955	Łoś	Proved Łoś's theorem forshadowing the transfer principle
1961, 1966	Abraham Robinson	Non-standard analysis
1977	Edward Nelson	Internal set theory

4.2 From du Bois-Reymond to Robinson

Du Bois-Reymond's investigations were in turn pursued further by such mathematicians as Emile Borel. In 1902, (Borel 1902, p. 35–36) cites Cauchy's definition of such "order of infinitesimal" [72] as inspiration for du Bois-Reymond's theory. Hardy (1910) provided a firm foundation for du Bois-Reymond's orders of infinity in 1910. Artin and Schreier (1926) developed the theory of real closed fields in 1926. Skolem (1934) constructed the first non-standard models of arithmetic in 1934. Hewitt's hyperreals of 1948 and Łoś's theorem of 1955 have already been discussed. In the 1960s, Robinson proposed an infinitesimal theory Robinson (1966) as an alternative to Weierstrassian epsilontic analysis. The evolution of infinitesimal from Cauchy to Robinson is summarized in Table 2.

Edward Nelson (1977) proposed an axiomatic theory parallel to Robinson's theory, see Subsect. 4.3. Ehrlich recently constructed an isomorphism of maximal hyperreal and surreal fields, resulting in a "unification of all numbers great and small" (Ehrlich).

[72] This definition appears in Oeuvres de Cauchy, série 2, tome 4, p. 181, corresponding to Cauchy's *Leçons sur le calcul différentiel* from 1829, appearing in a section entitled "Préliminaires"; see (Fisher 1981, p. 144).

4.3 Nelson

Edward Nelson (1977) proposed an axiomatic theory parallel to Robinson's theory. Nelson's axiomatisation, called Internal Set Theory (IST), takes the form of an enrichment of the Zermelo-Fraenkel set theory (ZFC) Fraenkel (1946). IST amounts to a more stratified axiomatisation for set theory, more congenial to infinitesimals.

In Nelson's system, the usual construction of $\mathbb{R}$, when interpreted with respect to the foundational background provided by IST, produces a number system already possessing entities behaving as infinitesimals.

In more detail, Nelson's approach is a re-thinking of the foundational material with a view to allowing a more stratified (hierarchical) number line. Thus, the canonical set theory, namely ZFC, is modified by the introduction of a unary predicate "standard". Then what is known as the usual construction of the "real" line produces a line that bears a striking resemblance to the Hewitt-Łoś-Robinson hyperreals.

To illustrate the power of Nelson's approach, we quote Alain Robert who starts his book on IST Robert (2003) with a homage to Leonard Euler:

> Here is how [Euler] deduces the expansion of the cosine function [...] He starts from the de Moivre formula
>
> $$\begin{aligned}\cos nz &= \frac{1}{2}\left[(\cos z + i\sin z)^n + (\cos z - i\sin z)^n\right]\\ &= \cos^n z - \frac{n(n-1)}{1\cdot 2}\cos^{n-2} z\sin^2 z\\ &\quad + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\cos^{n-4} z\sin^4 z + \cdots\end{aligned}$$
>
> and writes [...]
>
> *sit arcus z infinite parvus; erit* $\cos\cdot z = 1,\ \ \sin\cdot z = z$*; sit autem n numerus infinite magnus, ut sit arcus nz finitae magnitudinis, puta $nz = v$*
>
> $$\cos\cdot v = 1 - \frac{v^2}{2!} + \frac{v^4}{4!}\text{—etc.}$$

In the context of internal set theory, this argument becomes fully rigorous.

5 Conclusion

As we have seen, Cauchy experimented with a range of foundational approaches, including infinitesimal methodologies. He anticipated a number of mathematical developments that occurred decades later. One such development was the study of the asymptotic behavior and rates of growth of functions, culminating in the construction of infinitesimal-enriched continua by du Bois-Reymond, Emile Borel, G. H. Hardy, and others. Another development anticipated by Cauchy was the Dirac delta function, of which Cauchy gave an infinitesimal definition. Similarly, Björling and Cauchy both articulated single-variable definitions of stronger forms of continuity and convergence (today called *uniform*), in the context of infinitesimal-enriched continua. One avenue for further exploration is Cauchy's infinitesimal approach to the degrees of contact among curves. [73] Such visionary anticipations in Cauchy

[73] Cauchy does not hesitate to characterize the center of curvature as the meeting point of a pair of infinitely close normals (Cauchy 1826, p. 91).

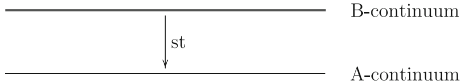

Fig. 1 Thick-to-thin: taking standard part (the thickness of the *top line* is merely conventional)

are studiously obfuscated by scholars sporting protective blinders, conditioned by a conceptual framework stemming from a nominalistic reconstruction of analysis set in motion by the "great triumvirate" of Cantor, Dedekind, and Weierstrass. According to the rules of such feedback-style ahistory, to borrow Grattan-Guinness's term, [74] Cauchy's infinitesimals are subjected to an automated translation to limits, the latter being promoted to first fiddle, Cauchy's explicit statements to the contrary notwithstanding (see Subsect. 3.1). Yet, many historians have refused to toe the triumvirate line. Cauchy's anticipations are highlighted in insightful studies by Freudenthal (1971), Robinson (1966), Sinaceur (1973), Lakatos (1978), Cleave (1979), Cutland et al. (1988), Medvedev (1987), Laugwitz (1989), Sad et al. (2001), Bråting (2007), and others.

It bears pointing out that the the calculus of Newton and Leibniz was not the same as ours, since they did not have a continuum that lives up to modern standards. Yet historians routinely attribute the invention of the calculus to them, implicitly taking it for granted that the calculus of Newton and Leibniz can be interpreted in terms of modern calculus, and vice versa. Isn't it time we applied such bi-interpretability to Cauchy's infinitesimals, as well?

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Appendix A: Rival Continua

A Leibnizian definition of the derivative as the infinitesimal ratio

$$\frac{\Delta y}{\Delta x},$$

whose logical weakness was criticized by Berkeley, was modified by A. Robinson by exploiting a map called *the standard part*, denoted "st", from the finite part of a B-continuum (for "Bernoullian"), to the A-continuum (for "Archimedean"), as illustrated in Fig. 1. [75]

We will denote such a B-continuum by a new symbol $\mathrm{I\!I\!R}$. We will also denote its finite part, by

$$\mathrm{I\!I\!R}_{<\infty} = \{x \in \mathrm{I\!I\!R} : \ |x| < \infty\}\,;$$

namely, the difference $\mathrm{I\!I\!R} \setminus \mathrm{I\!I\!R}_{<\infty}$ consists of the inverses of nonzero infinitesimals. The map "st" sends each finite point $x \in \mathrm{I\!I\!R}$, to the real point $\mathrm{st}(x) \in \mathbb{R}$ infinitely close to x:

$$\begin{array}{c} \mathrm{I\!I\!R}_{<\infty} \\ \big\downarrow{\scriptstyle \mathrm{st}} \\ \mathbb{R} \end{array}$$

[74] See main text at footnote 65.

[75] In the context of the hyperreal extension of the real numbers, the map "st" sends each finite point x to the real point $\mathrm{st}(x) \in \mathbb{R}$ infinitely close to x. In other words, the map "st" collapses the cluster of points infinitely close to a real number x, back to x. A comparative study of continua from a predicative angle may be found in Feferman (2009).

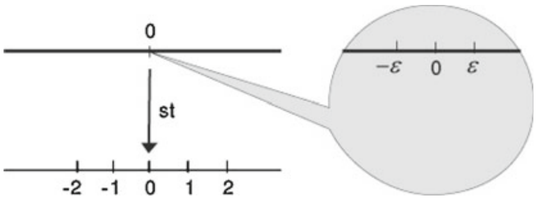

Fig. 2 Zooming in on infinitesimal ϵ

Robinson's answer to Berkeley's *logical criticism* (see Sherry (1987)) is to define the derivative of $y = f(x)$ as

$$f'(x) = \mathrm{st}\left(\frac{\Delta y}{\Delta x}\right),$$

rather than the infinitesimal ratio $\Delta y/\Delta x$ itself, as in Leibniz. "However, this is a small price to pay for the removal of an inconsistency" (Robinson 1966, p. 266).

We illustrate the construction by means of an infinite-resolution microscope in Fig. 2.

Note that both the term "hyper-real field", and an ultrapower construction thereof, are due to E. Hewitt in 1948, see (Hewitt 1948, p. 74). The transfer principle allowing one to extend every first-order real statement to the hyperreals, is due to Łoś in 1955, see Łoś (1955). Thus, the Hewitt-Łoś framework allows one to work in a B-continuum satisfying the transfer principle.

Hewitt's construction of hyper-real fields has roots in functional analysis, including works by Gelfand and Kolmogoroff (1939). In 1990, Hewitt reminisced about his

> efforts to understand the ring of all real-valued continuous [not necessarily bounded] functions on a completely regular T_0-space. I was guided in part by a casual remark made by Gel'fand and Kolmogorov [76] [...] Along the way I found a novel class of real-closed fields that superficially resemble the real number field and have since become the building blocks for nonstandard analysis. I had no luck in talking to Artin about these hyperreal fields, though he had done interesting work on real-closed fields in the 1920s. (My published 'proof' that hyperreal fields are real-closed is false: John Isbell earned my gratitude by giving a correct proof some years later.) [...] My ultra-filters also struck no responsive chords. Only Irving Kaplansky seemed to think my ideas had merit. My first paper on the subject was published only in 1948 (Hewitt 1990)

(Hewitt goes on to detail the influence of his 1948 text). Here Hewitt is referring to Isbell's 1954 paper Isbell (1954), proving that Hewitt's hyper-real fields are real closed. Note that a year later, Łoś (1955) proved the general transfer principle for such fields, implying in particular the property of being real closed.

To elaborate on the ultrapower construction of the hyperreals, let $\mathbb{Q}^{\mathbb{N}}$ denote the space of sequences of rational numbers. Let $\left(\mathbb{Q}^{\mathbb{N}}\right)_C$ denote the subspace consisting of Cauchy sequences. The reals are by definition the quotient field

$$\mathbb{R} := \left(\mathbb{Q}^{\mathbb{N}}\right)_C \Big/ \mathcal{F}_{null},$$

where the ideal $\mathcal{F}_{null}$ contains all the null sequences. Meanwhile, an infinitesimal-enriched field extension of $\mathbb{Q}$ may be obtained by forming the quotient

$$\mathbb{Q}^{\mathbb{N}} \Big/ \mathcal{F}_u,$$

see Fig. 3. Here a sequence $\langle u_n : n \in \mathbb{N}\rangle$ is in $\mathcal{F}_u$ if and only if the set

[76] Here Hewitt cites Gelfand and Kolmogoroff (1939).

Fig. 3 An intermediate field $\mathbb{Q}^{\mathbb{N}} \big/ \mathcal{F}_u$ is built directly out of $\mathbb{Q}$

$$\begin{tikzcd}
 & \left(\mathbb{Q}^{\mathbb{N}} \big/ \mathcal{F}_u\right)_{<\infty} \arrow[r, hook] \arrow[d, "\mathrm{st}"] & \mathrm{I\!I\!R}_{<\infty} \arrow[d, "\mathrm{st}"] \\
\mathbb{Q} \arrow[ur, hook] \arrow[r] & \mathbb{R} \arrow[r, "\simeq"] & \mathbb{R}
\end{tikzcd}$$

Table 3 Modern mathematical implementation of 17th century heuristic concepts

Heuristic concept	Mathematical implementation
Adequality	Standard part function
Law of continuity	Transfer principle
Infinitesimal-enriched continuum	Hyperreal number line

$$\{n \in \mathbb{N} : u_n = 0\}$$

is a member of a fixed ultrafilter. [77] To give an example, the sequence $\left\langle \frac{(-1)^n}{n} \right\rangle$ represents a nonzero infinitesimal, whose sign depends on whether or not the set $2\mathbb{N}$ is a member of the ultrafilter. To obtain a full hyperreal field, we replace $\mathbb{Q}$ by $\mathbb{R}$ in the construction, and form a similar quotient

$$\mathrm{I\!I\!R} := \mathbb{R}^{\mathbb{N}} \big/ \mathcal{F}_u.$$

A more detailed discussion of the ultrapower construction can be found in Davis (1977). See also Błaszczyk (2009) for some philosophical implications. More advanced properties of the hyperreals such as saturation were proved later, see Keisler (1994) for a historical outline. A helpful "semicolon" notation for presenting an extended decimal expansion of a hyperreal was described by Lightstone (1972). See also Roquette (2010) for infinitesimal reminiscences. A discussion of infinitesimal optics is in Stroyan (1972), Keisler (1986), Tall (1980), and Magnani and Dossena (2005), Dossena and Magnani (2007). Applications of the B-continuum range from aid in teaching calculus (Ely 2010; Katz and Katz 2010a,b; Tall 1991, 2009) to the Bolzmann equation, see Arkeryd (1981, 2005).

An interesting elementary example of a B-continuum is the ring of dual numbers, that is, numbers of the form $a + b\delta$, where $\delta^2 = 0$. It is useful, for example, in the theory of linear algebraic groups and linear Lie groups, and enable a quick and transparent computation of Lie algebras of groups like $SO_3(\mathbb{R})$.

Recently, Giordano (2010a,b, 2011) introduced a systematic way of enriching the reals by nilpotent infinitesimals, referring to the resulting structure as *the ring of Fermat reals*.

The transfer principle of the modern theory of infinitesimals is a mathematical implementation of Leibniz's heuristic law of continuity, see (Robinson, 1966, p. 266), and Laugwitz (1992). The standard part function is a mathematical implementation of Fermat's concept of adequality, see Table 3.

[77] An ultrafilter on $\mathbb{N}$ can be thought of as a way of making a systematic choice, between each pair of complementary infinite subsets of $\mathbb{N}$, so as to prescribe which one is "dominant" and which one is "negligible". Such choices have to be made in a coherent manner, e.g., if a subset $A \subset \mathbb{N}$ is negligible then any subset of A is negligible, as well. The existence of ultrafilters was proved by Tarski (1930), see (Keisler, 2008, Theorem 2.2). See a related remark about P-points in footnote 69.

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